Lecture/ E-learning notes

Course Code : 17CE3101

Course Title : Structural Analysis-II

Class : III B. Tech I Sem

Branch : Civil Engineering

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UNIT-I
INDETERMINATE STRUCTURES

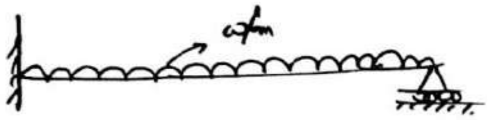
Structures which cannot be analysed with the equations of statics alone are called Statically Indeterminate Structures. (or) Hyper static structures.

→ The no. of unknown forces is greater than the no. of equilibrium equations

→ For complete analysis, additional equations based on conditions of compatibility & consistent displacements (or) deformations shall be used

Ex: Fixed beams, Continuous Beams, propped beams, portal frames.

Ex



Classification of structures

1) Skeletal structures: Structures which are idealized to a series of straight & curved lines

Ex: Roof trusses, Building frames.

2) Surface structures: Structures which can be idealized to a plane & curved surfaces.

Ex: Slabs & shells

3) Solid structures: Structures which can neither be idealized to a skeleton nor to surface surface.

Ex: Massive foundation.

Classification of skeletal structures

a) Based on type of Joint —
- Pin Jointed frames
- Rigid

1) Pin Jointed frames

→ Members are connected by means of pin Joints

→ frames support the loads by Axial forces only.

2) Rigid Jointed frames

→ The Joints are of rigid Jointed frames are assumed to be rigid so that the angles b/w the members meeting @ a Joint remains unchanged.

(b) Based on Dimensions

i) plane frames

→ All members of the plane frame as well as external loads are assumed to be in one plane

→ plane frames ─┬→ Pin Jointed plane frame → Axial forces only
                └→ Rigid " " " → Axial, shear, Bending

* If the Loading is in its own plane, cross section of member is subjected to '3' internal forces (1 AF, 1SF, 1BM)

* If the Loading is away from the plane, (1AF, 1SF, 1BM, 1TM)

(ii) Space frames

→ All members do not lie in a same plane.

→ Space frames ─┬→ Pin Jointed Space frame → Axial forces only
                └→ Rigid " " " → AF, SF, BM, TM.

* At any c/s of a member of a "Skeletal space structure" they are 6 Internal force components.

- → 1 Axial force.
- → 2 shear forces (f_x, f_y)
- → 1 Twisting moments
- → 2 Bending moments (M_x, M_y)

Equations of static Equilibrium

i) for a plane frame

$$\Sigma f_x = \Sigma f_y = \Sigma M_z = 0$$
(&)
$$\Sigma H = \Sigma V = \Sigma M = 0$$

} r = 3 ↳ No of eq. equations.

ii) for a space frame $\Sigma f_x = \Sigma f_y = \Sigma f_z = \Sigma M_x = \Sigma M_y = \Sigma M_z = 0$ } r = 6.

Degree of static indeterminacy (or) Degree of Redundancy

$$D_s = D_{se} + D_{si}$$

D_{se} = External indeterminacy   R = No. of reaction components

= R − 6, for space frame

= R − 3 for plane frame.

D_{si} = Internal indeterminacy

= m − (2j − 3) for pin jointed plane frame

= m − (3j − 6) for pin " space "   J = No. of joints

= 3c for rigid jointed plane ,   c = No. of cuts req. for obtaining a open configuration

= 6c for " " space ,   = No. of closed boxes

(or)

Simplied formula

D_s = (m + R) − 2j, for a pin jointed plane frame

= (m + R) − 3J, " " space "

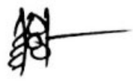
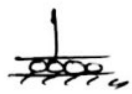
= (3m + R) − 3J, " Rigid " plane ,

= (6m + R) − 6J, " " " space "

m = No. of member forces

R = Reaction components

Reaction components (R)

Type of support	fig	Reactions (R)
a) Fixed support		3
b) Pinned/Hinged "		2
c) Roller support		1
d) Free end		0
e) Vertical shear hinge. Support/ Guided roller		2
f) Horizontal shear hinge		2

Releases (Additional equations)

i) Internal pin/Hinge

→ A pin provided anywhere in the structure can transmit moment from one part of structure to other part and provides '1' additional condition, $\Sigma M = 0$

ii) Internal Link

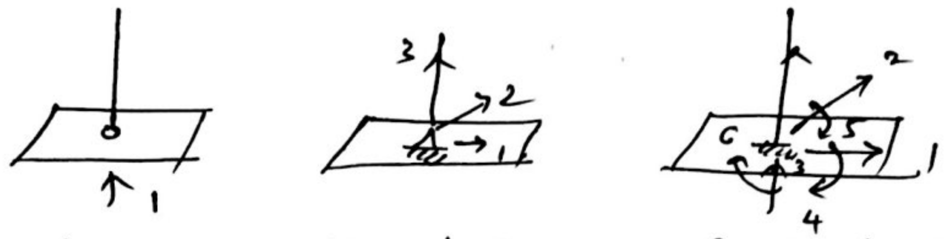
→ A link (consisting of short bar with pin @ each end) provided anywhere in the structure which is incapable of transmitting movement & Horizontal force from one part to other and thus provide '2' Additional equations.

$$\Sigma M = 0, \Sigma H = 0$$

Rigid Joint - Space frame

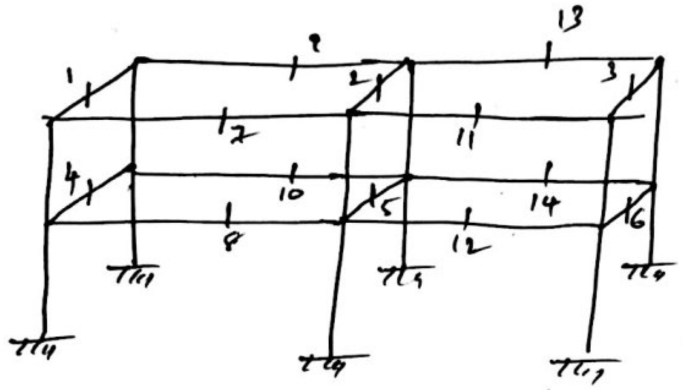
→ At any point they are six components of force (3 forces + 3 moments) in space frames

Reaction components @ supports



Roller end - 1 Hinged - 3 Fixed end - 6

Ex:



No. of reaction components = $6 \times 6 = 36$.

No. of cuts made to get determinate = 14,

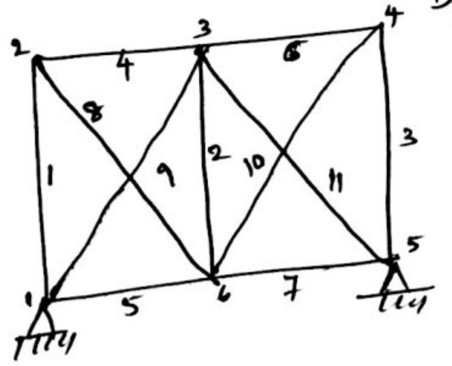
No. of unknowns cuts = $14 \times 6 = 84$

No. of portions = 6

No. of independent equations = $6 \times 6 = 36$

$\therefore$ Degree of indeterminacy = $36 + 84 - 36 = 84$.

Pin Jointed frames



$D_{sp} = m - 2j - 2$

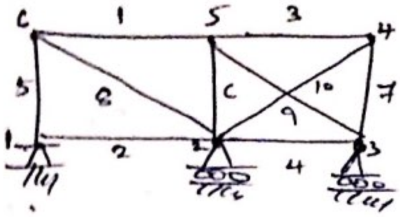
$= 11 - \{2(6) - 3\}$

$= 11 - 9 = 2$

$D_{se} = 4 - 3 = 1$

$D = 1 + 2 = 3$.

2)



$D_{se} = 4 - 3 = 1$

$D_{si} = m - (2j - r)$

$= 10 - \{2(6) - 3\}$

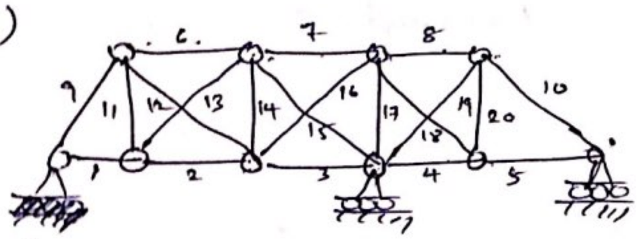
$= 10 - 9 = 1$

$\boxed{D_s = 1 + 1 = 2}$

Note: Basic perfect frame is a Δle, Hence in Second panel 1 extra redundant member. Hence $D_{si} = 1$ from simple observation

$\therefore D_s = D_{se} + D_{si} = 1 + 1 = 2$

3)



$D_{se} = 2 + 1 + 1 = 4$

$D_{se} = 4 - 3 = 1$

$D_{si} = m - (2j - r)$

$= 20 - \{2(10) - 3\}$

$= 20 - 17 = 3$

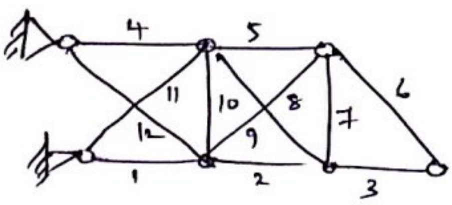
$\boxed{D_s = 1 + 3 = 4}$

Note: $D_{se} = 4 - 3 = 1$

$D_{si} = 3$ (extra redundant members in central 3 spans)

$\boxed{D_{se} = 1 + 3 = 4}$.

4)



$D_{se} = (2 + 2) - 3 = 1$

$D_{si} = m - (2j - r)$

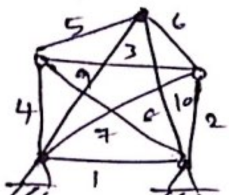
$= 12 - (2(7) - 3)$

$= 12 - 11 = 1$

$\boxed{D_s = 1 + 1 = 2}$

14-3.

5)



$D_{se} = 4 - 3 = 1$

$D_{si} = m - (2j - r)$

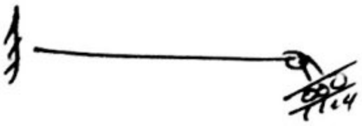
$= 10 - (2(5) - 3)$

$= 10 - 7 = 3$

$\boxed{D_{se} = 4}$

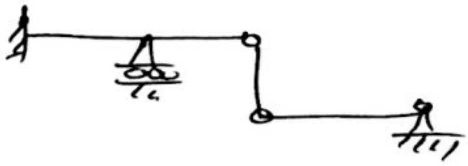
Assignment

1) Beams

i) 

ii) 

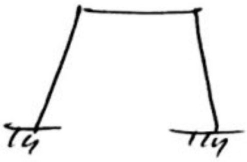
iii) 

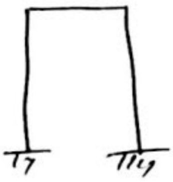
iv) 

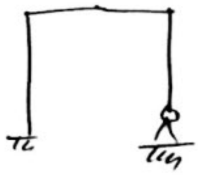
v) 

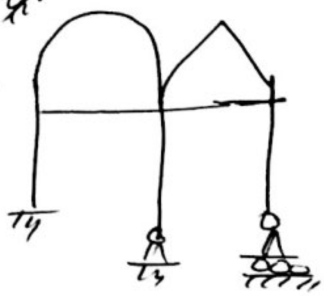
vi) 

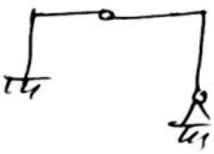
2) frames

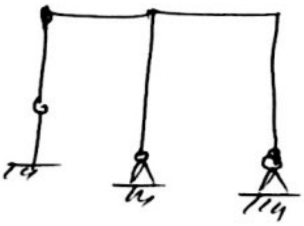
1) 

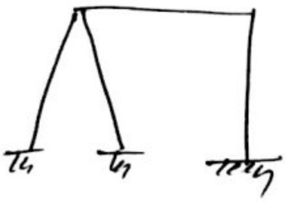
2) 

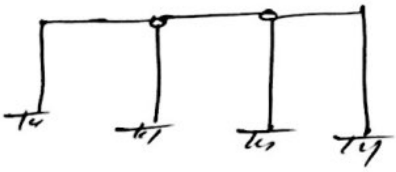
3) 

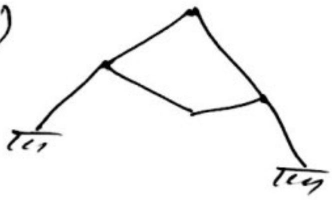
4) 

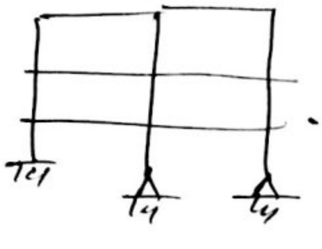
5) 

6) 

7) 

8) 

9) 

10) 

Kinematic indeterminacy

Kinematic indeterminacy (Degree of freedom) (D_k)

The no. of unknown joint displacements is called Degrees of freedom.

Kinematically indeterminate structure

→ If the displacement components of the joints of a skeletal structure cannot be determined by compatibility eq. alone, it is called kinematically indeterminate

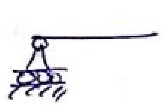
i.e; No. of unknown displacement components is greater than the no. of compatibility equations.

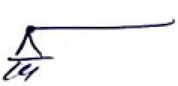
Degree of freedom for diff joints

Type of joint	Degree of freedom (D_k)
1) Pin joint of a plane frame	2 (Translations) $(\delta x, \delta y)$
2) pin joint of a Space frame	3 (Translations) $(\delta x, \delta y, \delta z)$
3) Rigid joint of a plane "	3 (1 rotation & 2 Translations) $[\delta x, \delta y, \theta]$
4) Rigid joint of a space "	6 (3 rotations & 3 translations)

Degree of freedom for diff. types of supports

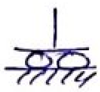
1) free end ——— 3 $[\delta x, \delta y, \theta]$

2) Roller Support  2 $[\theta, \delta x]$ Rotation, Translation.

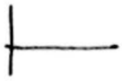
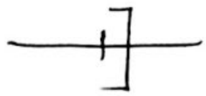
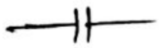
3) Hinged "  1 $[\theta]$ Rotation

4) fixed "  0

5) Vertical shear hinge  1 $[\delta y]$

6) Horizontal "  1 $[\delta x]$

Degree of freedom for typical joints

1)	Rigid joint		3	$(\delta x, \delta y, \theta)$
2)	Internal hinge in frame (Tangential)		4	$\delta x, \delta y, \theta_x, \theta_y$
3)	Internal hinge @ rigid joint		5	$\delta x, \delta y$, 3 rotations
4)	Internal hinge in beams		4	$\delta x, \delta y$, 2 Nos of rotations
5)	Horizontal shear release		4	2 Nos of δx, 1 δy, 1 rotation
6)	Vertical shear release		4	δx, 2 Nos of δy, 1 rotation

Rigid jointed plane frames $D_K = NJ - C$

where N = No. of degrees of freedom of each joint

J = No. of joints

C = No. of reaction components (R) if members are extensible

$= (m+R)$, if extension of members 'm' are neglected (Inextensible)

$\therefore$ $D_K = 3j - R$ Rigid jointed plane frame considering axial strains of members also

$D_K = 6J - R$ Rigid jointed space "

$D_K = 3j - (m+R)$ " " plane " Neglecting A.S.

$D_K = 6j - (m+R)$ " " space " " "

1) Kinematic indeterminacy of beam

$R = 2+1+1 = 4$

$D_k = 3j - R$

$= 3(3) - 4$

$= 9 - 4 = 5$

Neglecting axial deformation = $D_k = 5 - 2 = 3$

(a) $D_k = 5$

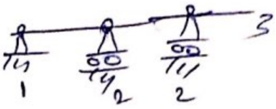
$D_k = 5 - 2 = 3$

2)

$R = 2+1+1 = 4$

$D_k = 3j - R = 3(4) - 4 = 12 - 4 = 8$

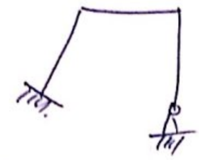
$D_k = 8 - 3 = 5$ [Neglecting axial strains]



(a) $D_k = 5 + 3 = 8$

$D_k = 8 - 3 = 5$ [N.A]

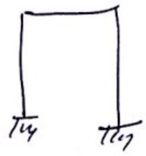
3)



$D_k = 3J - R = 3(4) - 5 = 7$

$D_k = 7 - 3 = 4.$ [N.A]

5)



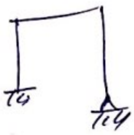
$D_k = 3J - R$

$= 3(4) - 6$

$= 12 - 6 = 6$

$D_k = 6 - 3 = 3$ [N.A].

6)



$D_k = 3J - R$

$= 3(4) - 5$

$= 7$

$D_k = 7 - 3 = 4$ [N.A]

4) Assuming column as inextensible.

$D_k = 3J - R = 3(2) - 3 = 6$

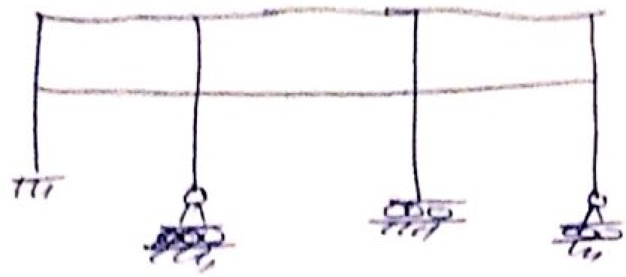
$D_k = 6 - 1 = 5$ [column]

(All members rigid)

~~Beam dir~~

$D_k = 6 - 2 = 4$

7)



$D_k = 3J - R$

$= 3(12) - 7 = 29$ [considered A.D]

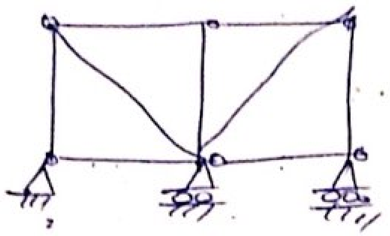
$D_k = 29 - 14 = 15$ [N.A.D]

Pin jointed plane frames

$D_k = 2J - R$ [pin jointed plane frame]

$= 3J - R$ [Pin " Space "]

1)



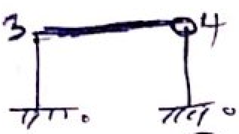
[Only Axial deformations are considered]

$D_k = 2J - R = 2(6) - 4 = 8.$

~~D_k = ~~

Practices

1)



(7)

$D_k = 3J - R$

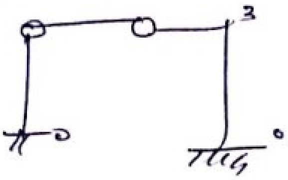
$= 3(4) - 6$

$= 12 - 6$

$= 6$

$D_k = 6 + 1 = 7$

2)



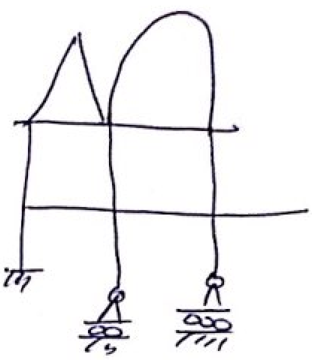
(11)

$D_k = 3(J) - R$

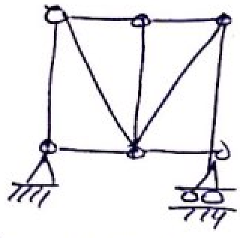
$= 3(5) - 6$

$= 15 - 6 = 9 + 2 = 11$

3)



4)

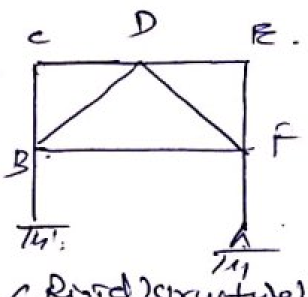


$D_k = 2J - R$

$= 2(6) - 3$

$= 12 - 3 = 9.$

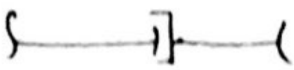
5)



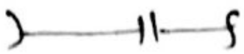
(Rigid structure)

$D_k = 3J - R$

$= 3(7) - 5 = \underline{16}$

iii) Horizontal shear release  → ($Q_H = 0$) → 1

→ can be used as a damper in machine foundations

iv) Vertical shear release 

No. of releases = 1 ($\Sigma V = 0$)

Beams

1) Determine the indeterminacy of various beams

for beams $R - 3$

1) 

No of equilibrium equations (r) = 3

No. of support reactions (R) = 3 + 2 = 5

∴ Degree of indeterminacy = No of support reactions − No. of equilibrium equation

= 5 − 3

= 2 → Indeterminate to second degree

(2) 

$r = 3$ → $D_I = 6 - 3 = 3$ → Third degree.

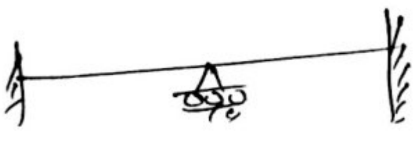
$R = 6$

(3) 

$r = 3$

$R = 2 + 1 + 1 = 4$

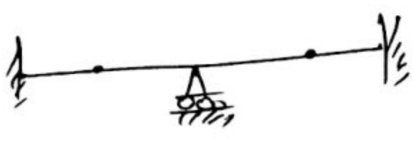
$D_I = 4 - 3 = 1$ → Indeterminate ~~first~~ degree

(4) 

$r = 3$

$R = 3 + 1 + 3 = 7$

$D_i = 7 - 3 = 4$

(5) 

$r = 3$

$R = 3 + 1 + 3 = 7$

Releases = 1 + 1 = 2

$D_i = 7 - 3 - 2$

$= 2$

• → pin/Hinge

→ (2)

(6) 

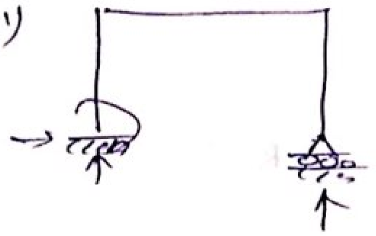
$r = 3$
$R = 3+1+2 = 6$
pin/hinge $= 1$

$D_i = 6-3-1$
$= 2$ (6)

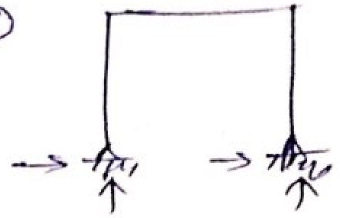
Frames → Rigid jointed

a) Plane frames ✓

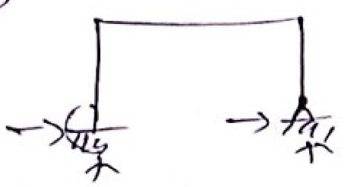
i) Single storey frames

1) 

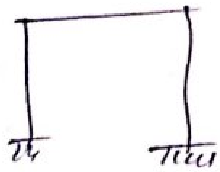
$r = 3$
$R = 3+1 = 4$
$D_i = 4-3 = 1$

(2) 

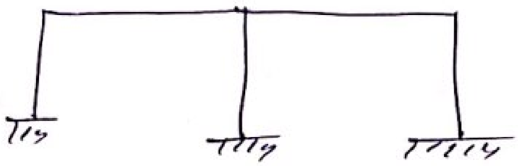
$r = 3$
$R = 2+2$
$= 4$
$D_i = 1$

(3) 

$r = 3$
$R = 3+2 = 5$
$D_i = 5-3 = 2$

4) 

$r = 3$
$R = 6$
$D_i = 6-3 = 3$

5) 

$r = 3$
$R = 3+3+3 = 9$
$D_i = 9-3 = 6.$

Multi storey frames

→ There will be internal indeterminacy also.

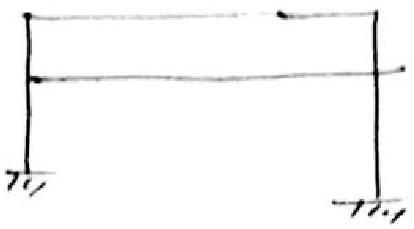
→ To find the total degree of indeterminacy in such frames, imagine several cuts in frames. ~~such that~~

→ In such a frame with cuts, No. of unknowns = No. of reaction components + 3 unknowns @ every cut

→ No. of equilibrium eq = 3 for each portion.

∴ ~~Deg~~ Total no. of independent static equilibrium eq are known.

1)



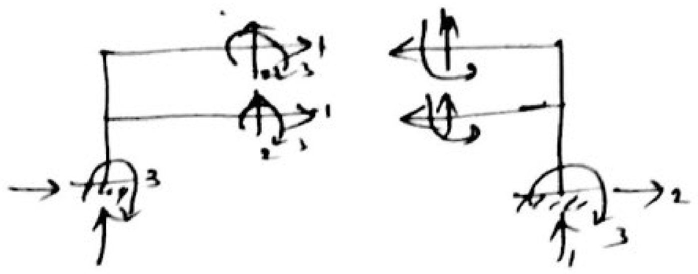
(b) $D_{se} = 6-3 = 3$

$D_p = 3c = 3\times1 = 3$

$D = 3+3 = 6$

Sol Let us take 2 cuts

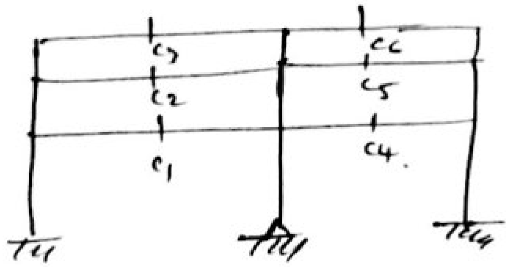
No. of reaction = 6 (3+3)
cuts = 2
No. of unknowns @ cuts = 3×2 = 6
Total unknowns = 6+6 = 12
Determinate = 2
= 3×2 = 6
Degree of = 12−6 = 6



D_p = No. of reactions − No. of equilibrium equation

No. of unknowns =

2)



No. of reaction components = 8 (3+2+3)

No. of cuts to make determinate = 6

No. of unknowns @ cuts = 3×6 = 18

Total no. of unknowns = 8+18 = 26

No. of determinate portions = 3

No. of independent equilibrium = 3×3 = 9

∴ Degree of indeterminacy D_I = 26−9 = 17

(or) $D_s = D_{se} + D_{si}$

$D_{se} = R - r = 8 - 3 = 5$

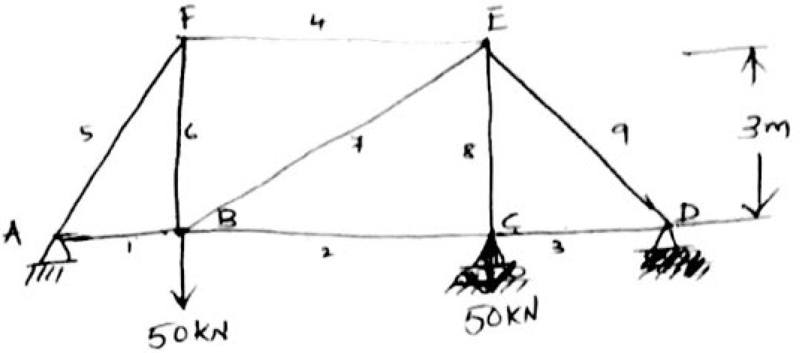
$D_{si} = 3c = 3\times4 = 12$

$D = D_{se} + D_{si}$
$= 5+12 = 17$

S.

2) Determine the forces in the members of the truss shown in fig below. The c/s area of each member is $A = 400 mm^2$ and $E = 2 \times 10^5 N/mm^2$

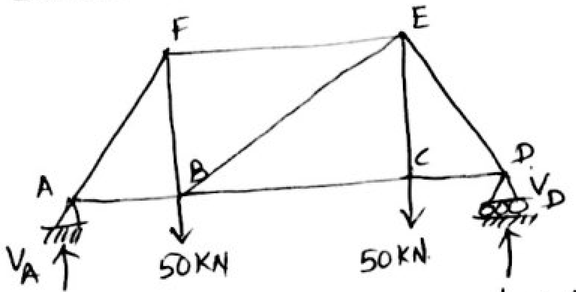
Sol



$D_{si} = m - (2j - r) = 9 - ((2(6) - 3) = 9 - 9 = 0$

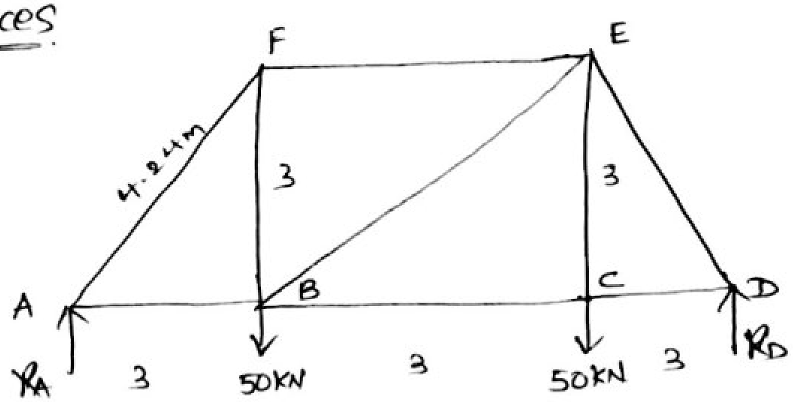
$D_{se} = 4 - 3 = 1 \rightarrow$ statically externally indeterminate to one degree

The horizontal reaction @ D is taken as redundant force & Hence restraint in this direction is released and basic determinate structure is obtained in figure below.



Now, the forces in the structure due to given loadings (P forces) and unit force in the direction R (K) forces are to be found

i) P-forces



$AF = \sqrt{3^2 + 3^2} = \sqrt{18}$
$= 4.24m$
$= 4242mm$

$\tan\theta = \frac{3}{3}$

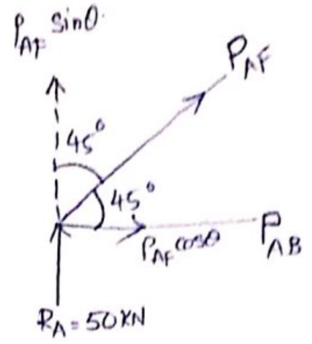
$\theta = 45°$

By symmetry $R_A = R_D = 50KN$

(or) Taking moments about D $\Rightarrow$ $R_A \times 9 - 50 \times 6 - 50 \times 3 = 0$

$9R_A = 450$

$R_A = \frac{450}{9} = 50\,KN$; $R_A + R_B = 100 \Rightarrow \boxed{R_B = 50\,KN}$



At Joint A

$\Sigma V = 0$

$\Rightarrow P_{AF}\sin\theta + R_A = 0$

$P_{AF} = -\frac{R_A}{\sin\theta} = -\frac{50}{\sin 45°} = -70.71\,KN$

$\boxed{P_{AF} = -70.71\,KN \text{ (Compressive)}}$

$\Sigma H = 0$

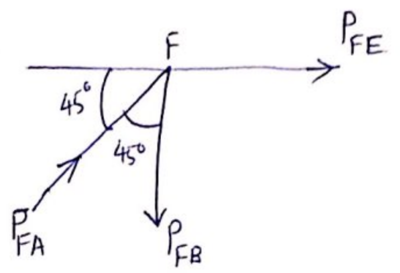
$\Rightarrow P_{AB} + P_{AF}\cos\theta = 0$

$P_{AB} + (-70.71)\cos 45° = 0$

$P_{AB} = 70.71\cos 45°$

$\boxed{P_{AB} = 50\,KN \text{ (Tensile)}}$

At Joint F



$\Sigma V = 0$

$\Rightarrow P_{FA}\cos 45° - P_{FB} = 0$
$(\uparrow)$ $(\downarrow)$

$-P_{FB} = -P_{FA}\cos 45°$

$P_{FB} = 70.71 \times \cos 45°$

$\boxed{P_{FB} = 50\,KN}$ (Tensile)

~~At Join~~ $\Sigma H = 0$

$\Rightarrow P_{FA}\cos 45° + P_{FE} = 0$
$(\rightarrow)$ $(\rightarrow)$

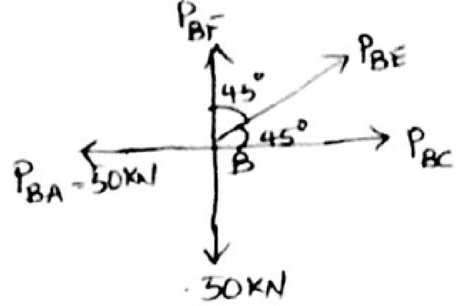
$P_{FE} = -P_{FA}\cos 45°$

$= -70.71\cos 45°$

$= -50\,KN$

$\therefore P_{FE} = 50\,KN$ (Comp)

At Joint B



$\Sigma V = 0$

$\Rightarrow \quad P_{BE}\cos 45^\circ + P_{BF} - 50 = 0$

(↑) (↑) (↓)

$P_{BE}\cos 45^\circ = -P_{BF} + 50$

$= -50 + 50 = 0$

$\boxed{P_{BE} = 0}$

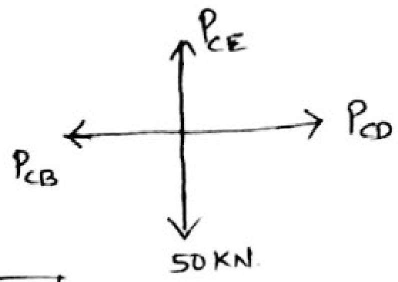
$\Sigma H = 0 \rightarrow \quad P_{BC} + P_{BE}\cos 45^\circ - P_{BA} = 0$

(→) (→) (←)

$P_{BC} + 0 - 50 = 0$

$\therefore \boxed{P_{BC} = 50\,KN \text{ (Tensile)}}$

At Joint C



$\Sigma V = 0 \Rightarrow \quad P_{CE} - 50 = 0 \Rightarrow \boxed{P_{CE} = 50\,KN \text{ (Tensile)}}$

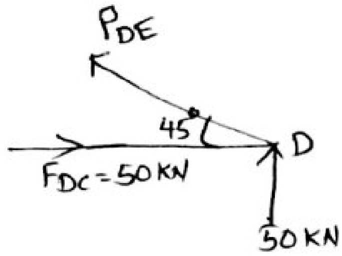
(↑) (↓)

$\Sigma H = 0 \Rightarrow \quad P_{CD} - P_{CB} = 0 \Rightarrow P_{CD} = P_{CB}$

(→) (←)

$\boxed{P_{CD} = 50\,KN \text{ (Tensile)}}$

At Joint D



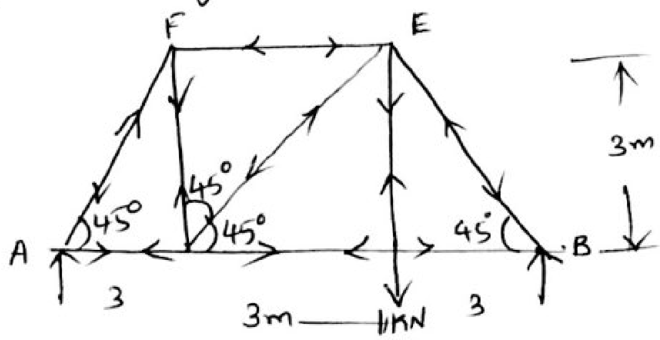
$\Sigma H = 0 \Rightarrow P_{DE}\cos 45^\circ + P_{DC} = 0$

(←) (←)

$P_{DE} = -\dfrac{P_{DC}}{\cos 45^\circ}$

$\boxed{P_{DE} = 70.71\,KN \text{ (comp)}}$

K-Forces Remove all the external loads and apply a unit vertical force @ joint 'c' of the truss.



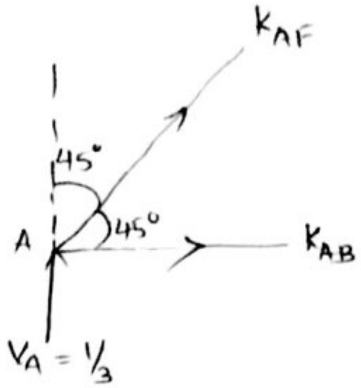
Taking moments about D $V_A \times 9 - 1 \times 3 = 0$

$9V_A = 3 \Rightarrow \boxed{V_A = \frac{1}{3} KN}$

Total load $= V_A + V_B$

$\Rightarrow V_B = 1 - 1/3 = \frac{2}{3} KN$

At Joint A



Initially assume all forces are to be tensile

$\Sigma V = 0$

$\Rightarrow K_{AF} \cos 45° + 1/3 = 0$
(↑) (↑)

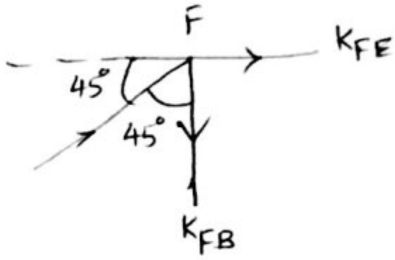
$K_{AF} \cos 45° = -1/3 \Rightarrow K_{AF} = \frac{-1}{3\cos 45°} = -0.471 KN$

$\boxed{K_{AF} = 0.471 KN \text{ (comp)}}$

$\Sigma H = 0 \Rightarrow K_{AF} \cos 45° + K_{AB} = 0$
(→) (→)

$-0.471 \times \cos 45° + K_{AB} = 0 \Rightarrow \boxed{K_{AB} = 0.33 KN}$ (Tensile)

At Joint F



$\Sigma H = 0 \Rightarrow K_{FA} \cos 45° + K_{FE} = 0$
(→)

$K_{FE} = -K_{FA} \cos 45° = -0.471 \times \cos 45°$
$= -0.333 KN$

$\boxed{K_{FE} = 0.333 KN \text{ (comp)}}$

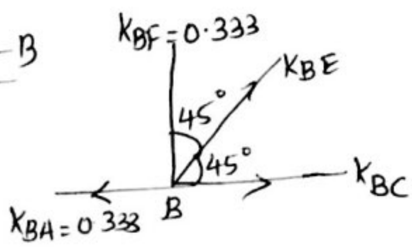
$\Sigma V = 0 \Rightarrow K_{FA} \cos 45° - K_{FB} = 0$
(↑) sin45° (↓)

$K_{FA} \cos 45° = K_{FB}$

$0.471 \cos 45° = K_{FB}$

$\boxed{K_{FB} = 0.333 KN \text{ (Tensile)}}$

At Joint B



$\Sigma V = 0 \Rightarrow K_{BE} \cos 45° + K_{BF} = 0$
(↑) (↑)

$K_{BE} = -\frac{K_{BF}}{\cos 45°} = \frac{-0.333}{\cos 45°} = -0.471$

$K_{BE} = 0.471$ (comp)

$\Sigma H = 0 \Rightarrow K_{BC} - K_{BA} + K_{BE}\cos 45° = 0$

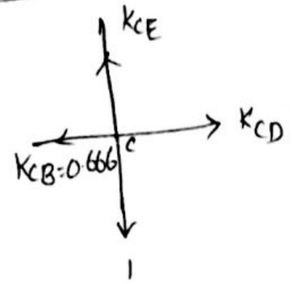
$(\rightarrow) \quad (\leftarrow) \quad (\rightarrow)$

$K_{BC} = K_{BA} - K_{BE}\cos 45°$

$= 0.333 - (-0.471)\cos 45°$

$K_{BC} = 0.666\,KN$ (Tensile)

At joint C



$\Sigma V = 0 \Rightarrow K_{CE} - 1 = 0$

$(\uparrow) \quad (\downarrow)$

$K_{CE} = 1\,KN$ (Tensile)

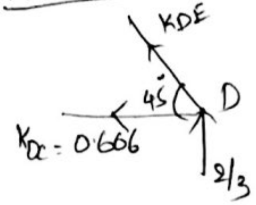
$\Sigma H = 0 \Rightarrow K_{CD} - K_{CB} = 0$

$(\rightarrow) \quad (\leftarrow)$

$K_{CD} = K_{CB} = 0.666$

$K_{CD} = 0.666\,KN$ (Tensile)

At joint D



$\Sigma H = 0 \Rightarrow K_{DE}\cos 45° + K_{DC} = 0$

$(\leftarrow) \quad (\leftarrow)$

$K_{DE} = -\frac{K_{DC}}{\cos 45°} = \frac{-0.666}{\cos 45} = -0.942\,KN$

$K_{DE} = 0.942\,KN$ (comp)

Horizontal deflection of joint 'D' is given by

$$\Sigma \frac{PKl}{AE} + R\,\Sigma \frac{K^2 l}{AE} = 0$$

$$R = -\frac{\Sigma \frac{PKl}{AE}}{\Sigma \frac{K^2 l}{AE}}$$

$=$

Calculation table

	Member	Length (mm)	Area (mm²)	P forces (KN)	K forces (KN)	$\frac{PKl}{AE}$	$\frac{K^2 l}{AE}$	S (P+RK)
1)	AF	4242	400	-70.71	-0.471	0.0017 ~~3.53419~~	~~0.0005~~ 1.12×10^{-5}	
2)	FE	3000	400	-50	-0.333	0.0006	4.15×10^{-6}	
3)	ED	4242	400	-70.71	-0.942	0.0035	4.7×10^{-5}	
4)	DC	3000	400	50	0.666	0.00124	1.66×10^{-5}	
5)	CB	3000	400	50	0.666	0.00124	1.66×10^{-5}	
6)	BA	3000	400	50	0.333	0.00062	4.15×10^{-6}	
7)	FB	3000	400	50	0.333	0.00062	4.15×10^{-6}	
8)	BE	4242	400	0	-0.471	00	1.76×10^{-5}	
9)	EC	3000	400	50	1.	0.0018	3.75×10^{-5}	
						0.01132	1.65×10^{-4}	

UNIT-II
INFLUENCE LINES

1.1 INTRODUCTION

Common sense tells us that when a load moves over a structure, the deflected shape of the structural will vary. In the process, we can arrive at simple conclusion that due to moving load position on the structure, reactions value at the support also will vary.

From the designer's point of view, it is essential to have safe structure, which doesn't exceed the limits of deformations and also the limits of load carrying capacity of the structure.

1.2 DEFINITIONS OF INFLUENCE LINE

In the literature, researchers have defined influence line in many ways. Some of the definitions of influence line are given below.

- An influence line is a diagram whose ordinates, which are plotted as a function of distance along the span, give the value of an internal force, a reaction, or a displacement at a particular point in a structure as a unit load move across the structure.
- An influence line is a curve the ordinate to which at any point equals the value of some particular function due to unit load acting at that point.
- An influence line represents the variation of the reaction, shear, moment, or deflection at a specific point in a member as a unit concentrated force moves over the member.

1.3 CONSTRUCTION OF INFLUENCE LINES

In this section, we will discuss about the construction of influence lines. Using any one of the two approaches (Figure 37.1), one can construct the influence line at a specific point P in a member for any parameter (Reaction, Shear or Moment). In the present approaches it is assumed that the moving load is having dimensionless magnitude of unity. Classification of the approaches for construction of influence lines is given in Figure below

Construction of Influence Lines

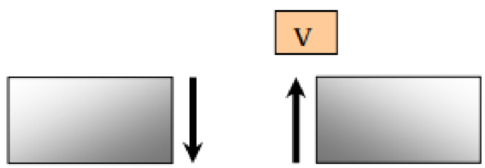
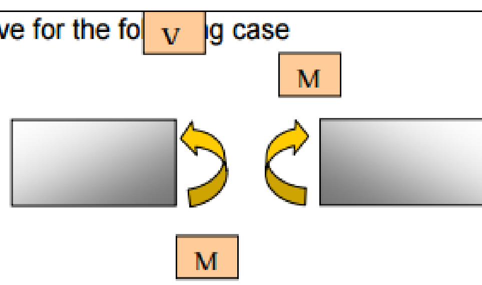
Tabulate Values   Influence Line-Equation

1.3.1 Tabulate Values

Apply a unit load at different locations along the member, say at ***x***. And these locations, apply statics to compute the value of parameter (reaction, shear, or moment) at the specified point. The best way to use this approach is to prepare a table, listing unit load at x versus the corresponding value of the parameter calculated at the specific point (i.e. Reaction R, Shear V or moment M) and plot the tabulated values so that influence line segments can be constructed.

1.3.2 Sign Conventions

Sign convention followed for shear and moment is given below.

Parameter	Sign for influence line
Reaction R	Positive at the point when it acts upward on the beam.
Shear V	Positive for the following case 
Moment M	Positive for the fo[illegible]g case 

1.3.3 Influence Line Equations

Influence line can be constructed by deriving a general mathematical equation to compute parameters (e.g. reaction, shear or moment) at a specific point under the effect of moving load at a variable position ***x***.

The above discussed both approaches are demonstrated with the help of simple numerical examples in the following paragraphs.

1.3.4 Getting Influence Line Equation

An influence line for a given function, such as a reaction, axial force, shear force, or bending moment, is a graph that shows the variation of that function at any given point on a structure due to the application of a unit load at any point on the structure.

1.4 SIMPLY SUPPORTED BEAMS

1.4.1 Load Categories

We can consider 5 categories of loads on beams

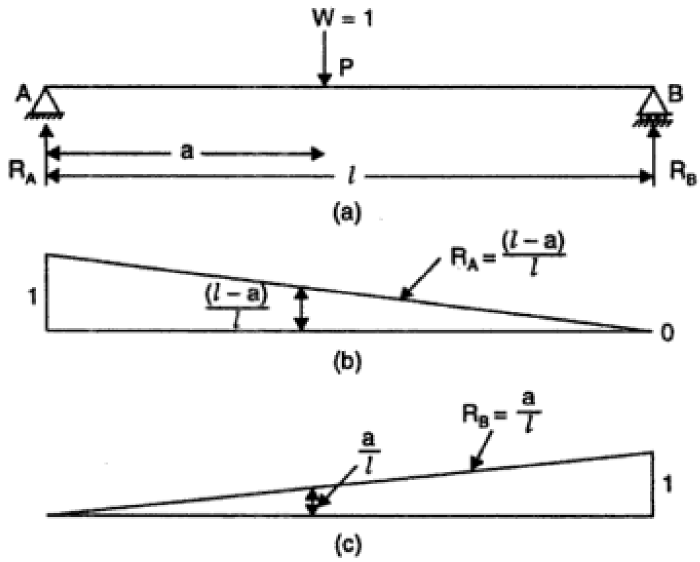
1. Concentrated Loads
    a. Single point load
    b. Two point load
    c. Multi point load
2. udl longer than the beam span
3. udl shorter than the beam span
4. Equivalent uniformly distributed load(EUDL)

1.5 CONCENTRATED LOADS

a) Single Point Load:

Reactions in a SSB

External forces like reactions are the easiest force components for which influence lines can be sketched easily.



Let us try to get IL for R_A for the beam AB in fig (a).Let a unit load act at P at a distance 'a' from A. Then R_A & R_B

$$R_A = \frac{(l-a)}{l}$$

$$R_B = \frac{a}{l}$$

ILD for Internal Shear & Bending moment in a SSB

Let us investigate the SF & BM at X at a distance 'x' from A. Let 'a' be the coordinate position of a unit load.

Shear force

For a < x

$$F_x = R_A - 1 = \frac{(l-a)}{l} - 1 = \frac{-a}{l}$$

For a > x

$$F_x = R_A = \frac{(l-a)}{l}$$

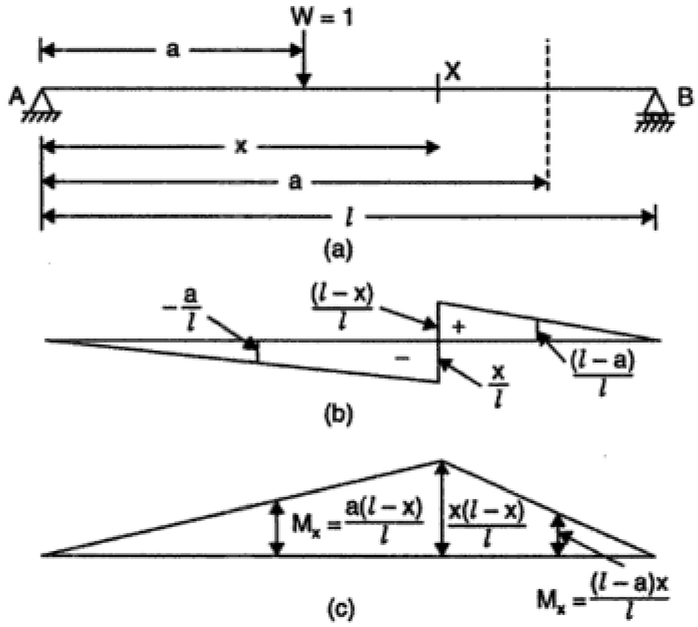
Bending moment

For a < x

$$M_x = R_B \times (l-x) = \frac{a}{l}(l-x)$$

For a > x

$$M_x = R_A X, \quad M_x = \frac{(l-a)}{l}x$$



Examples

1. Construct the influence line for the reaction at support B for the beam of span 10 m. The beam structure is shown in Figure below

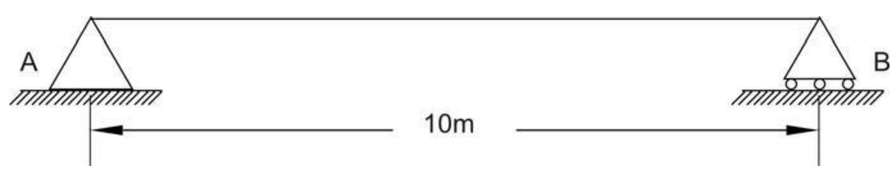


Figure : The beam structure

Solution:

As discussed earlier, there are two ways this problem can be solved. Both the approaches will be demonstrated here.

Tabulate values:

As shown in the figure, a unit load is places at distance x from support A and the reaction value R_B is calculated by taking moment with reference to support A. Let us say, if the load is placed at 2.5 m. from support A then the reaction R_B can be calculated as follows (Figure).

$$\Sigma M_A = 0 : R_B \times (10 - 1) \times 2.5 = 0 \Rightarrow R_B = 0.25$$

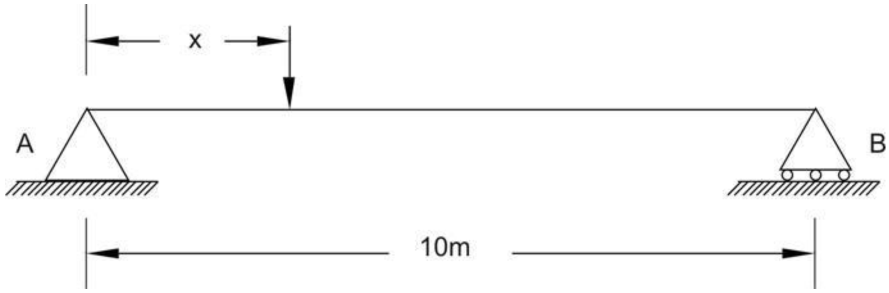


Figure 1 : The beam structure with unit load

Similarly, the load can be placed at 5.0, 7.5 and 10 m. away from support A and reaction R_B can be computed and tabulated as given below.

x	R_B
0	0.0
2.5	0.25
5.0	0.5
7.5	0.75
10	1

Graphical representation of influence line for R_B is shown in Figure 37.4.

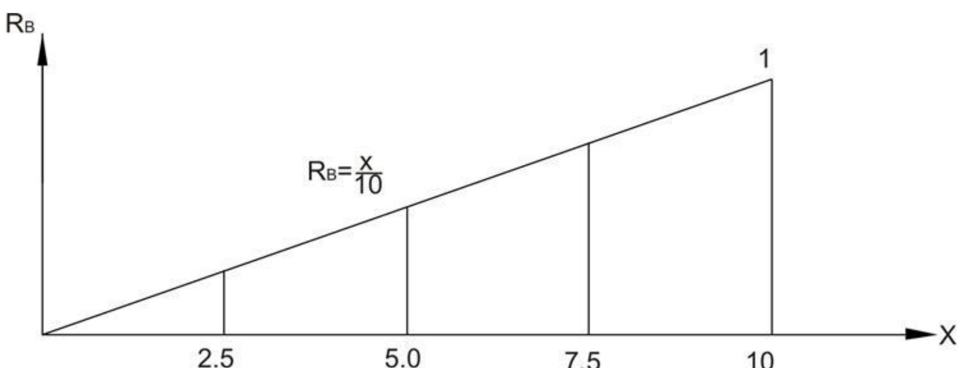


Figure 2: Influence line for reaction R_B.

Influence Line Equation:

When the unit load is placed at any location between two supports from support A at distance x then the equation for reaction R_B can be written as

$$\Sigma M_A = 0 : R_B \times (10 - x) = 0 \Rightarrow R_B = x/10$$

The influence line using this equation is shown in Figure 2.

2. Construct the influence line for support reaction at B for the given beam as shown in below.

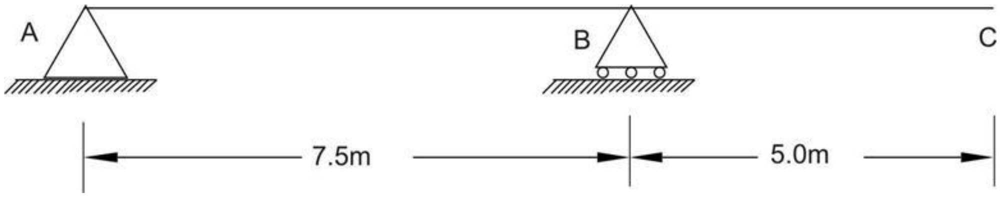


Figure: The overhang beam structure

Solution:

As explained earlier in example 1, here we will use tabulated values and influence line equation approach.

Tabulate Values:

As shown in the figure, a unit load is places at distance x from support A and the reaction value R_B is calculated by taking moment with reference to support A. Let us say, if the load is placed at 2.5 m. from support A then the reaction R_B can be calculated as follows.

$$\Sigma M_A = 0 : R_B \text{ x } (7.5 - 1) \text{ x } 2.5 = 0 \Rightarrow R_B = 0.33$$

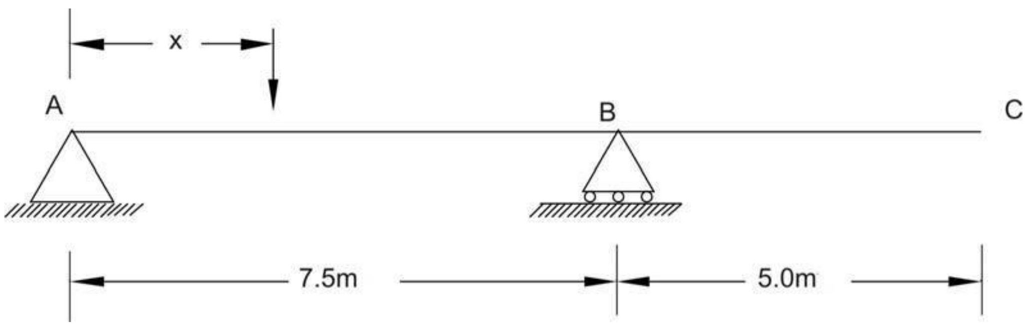


Figure 1: The beam structure with unit load

Similarly one can place a unit load at distances 5.0 m and 7.5 m from support A and compute reaction at B. When the load is placed at 10.0 m from support A, then reaction at B can be computed using following equation.

$$\Sigma M_A = 0 : R_B \text{ x } (7.5 - 1) \text{ x } 10.0 = 0 \Rightarrow R_B = 1.33$$

Similarly a unit load can be placed at 12.5 and the reaction at B can be computed. The values of reaction at B are tabulated as follows.

x	R_B
0	0.0
2.5	0.33
5.0	0.67
7.5	1.00
10	1.33
12.5	1.67

Graphical representation of influence line for R_B is shown in Figure 2.

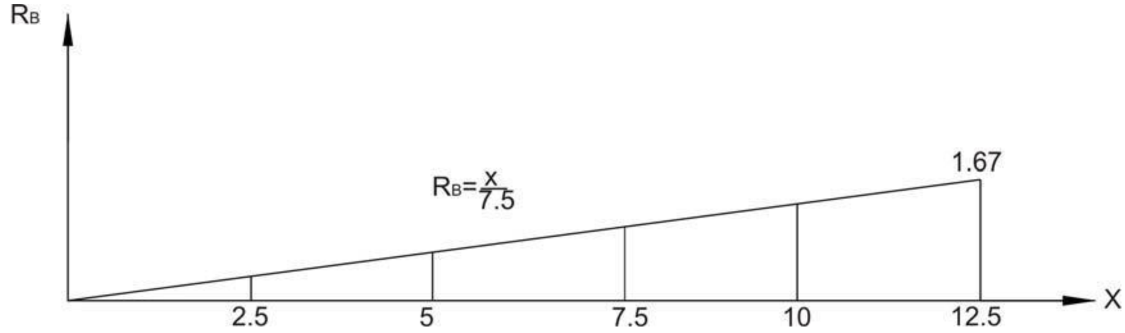


Figure 2: Influence for reaction R_B.

Influence line Equation:

Applying the moment equation at A (Figure 37.6),

$\Sigma M_A = 0 : R_B \times (7.5 - 1) \times x = 0 \Rightarrow R_B = x/7.5$

The influence line using this equation is shown in Figure 2.

3. Construct the influence line for shearing point C of the beam (Figure 37.8)

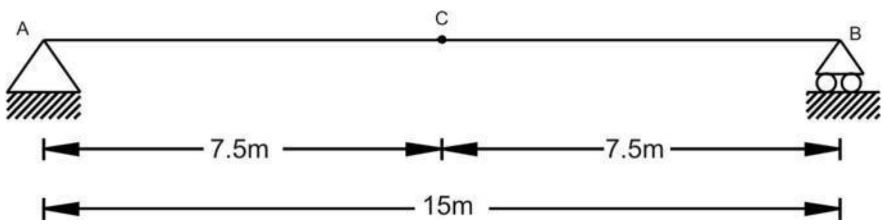


Figure: Beam Structure

Solution:

Tabulated Values:

As discussed earlier, place a unit load at different location at distance x from support A and find the reactions at A and finally computer shear force taking section at C. The shear force at C should be carefully computed when unit load is placed before point C (Figure 1) and after point C (Figure 2). The resultant values of shear force at C are tabulated as follows.

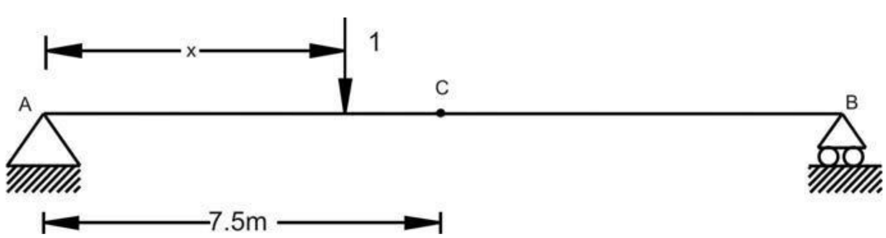


Figure 1: The beam structure – a unit load before section

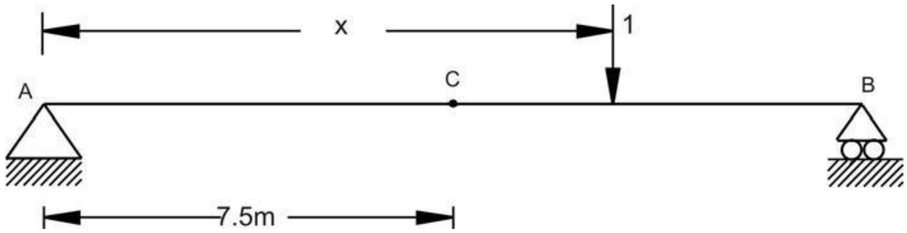


Figure 2 : The beam structure - a unit load before section

X	V_C
0	0.0
2.5	-0.16
5.0	-0.33
7.5(-)	-0.5
7.5(+)	0.5
10	0.33
12.5	0.16
15.0	0

Graphical representation of influence line for V_C is shown in Figure 3.

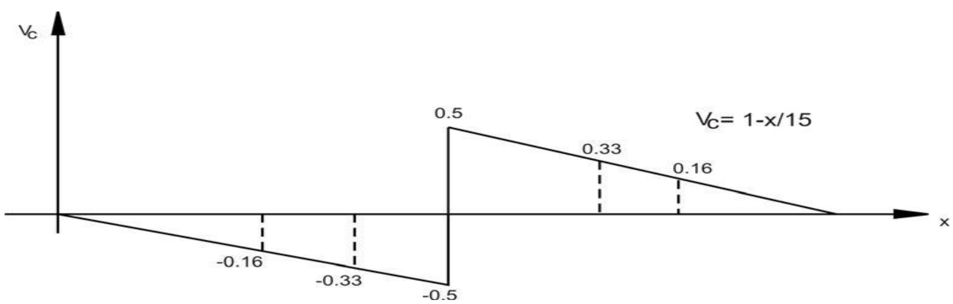


Figure 3: Influence line for shear point C

Influence line equation:

In this case, we need to determine two equations as the unit load position before point C (Figure 4) and after point C (Figure 5) will show different shear force sign due to discontinuity. The equations are plotted in Figure 3.

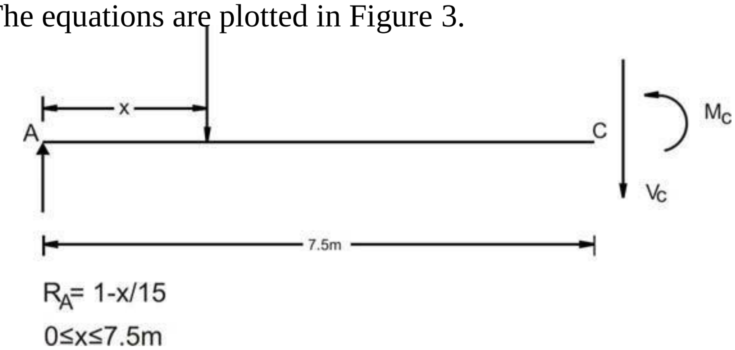


Figure 4: Free body diagram – a unit load before section

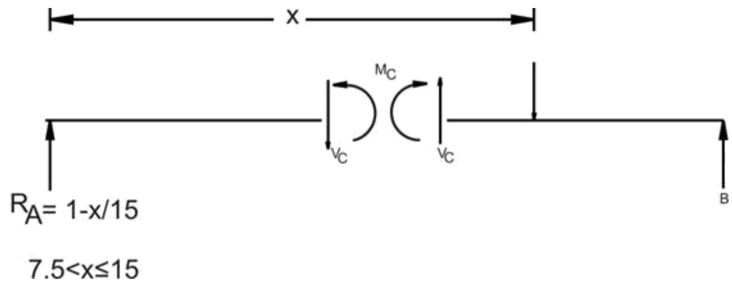


Figure 5: Free body diagram – a unit load after section

Influence Line for Moment:

Like shear force, we can also construct influence line for moment.

4. Construct the influence line for the moment at point C of the beam shown in Figure

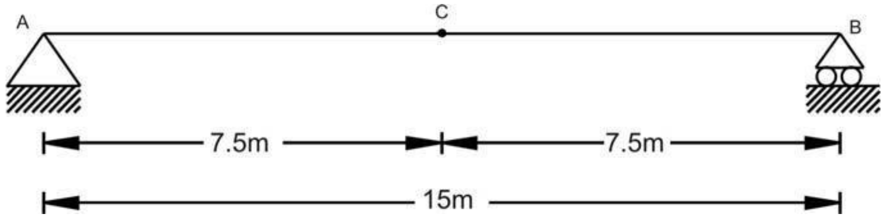


Figure: Beam structure

Solution:

Tabulated values:

Place a unit load at different location between two supports and find the support reactions. Once the support reactions are computed, take a section at C and compute the moment. For example, we place the unit load at x=2.5 m from support A (Figure 1), then the support reaction at A will be 0.833 and support reaction B will be 0.167. Taking section at C and computation of moment at C can be given by

$\Sigma M_c = 0 : - M_{c} + R_B \times 7.5 - = 0 \Rightarrow - M_{c} + 0.167 \times 7.5 - = 0 \Rightarrow M_c = 1.25$

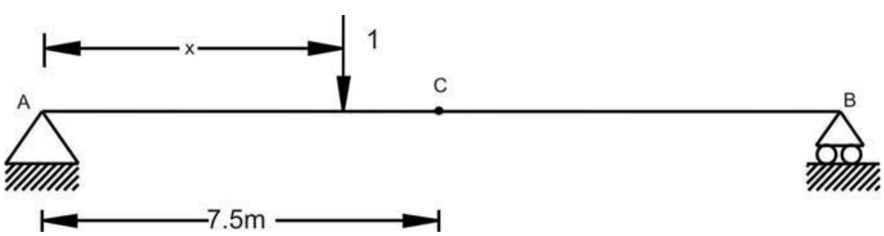
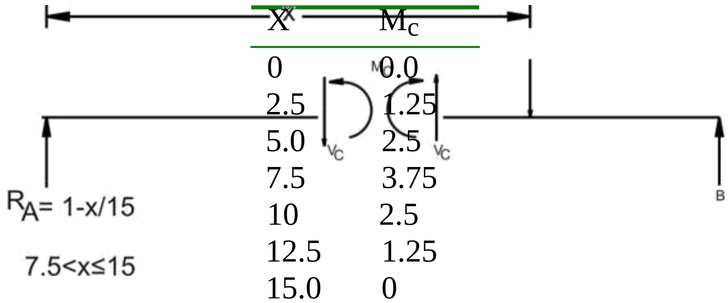


Figure 1: A unit load before section

Similarly, compute the moment M_c for difference unit load position in the span. The values of Mc are tabulated as follows.



X	M_C
0	0.0
2.5	1.25
5.0	2.5
7.5	3.75
10	2.5
12.5	1.25
15.0	0

Graphical representation of influence line for M_C is shown in Figure 2.

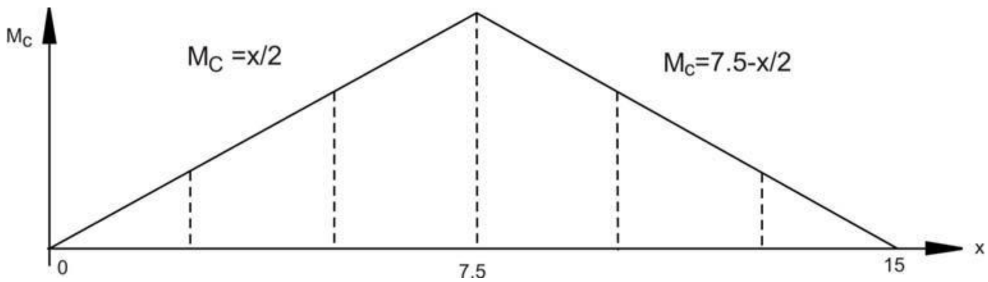


Figure 2: Influence line for moment at section C

Influence Line Equations:

There will be two influence line equations for the section before point C and after point C.

When the unit load is placed before point C then the moment equation for given Figure 3 can be given by

$\Sigma M_C = 0 : M_C + 1(7.5 - x) - (1 - x/15)x7.5 = 0 \Rightarrow M_C = x/2$, where $0 \le x \le 7.5$

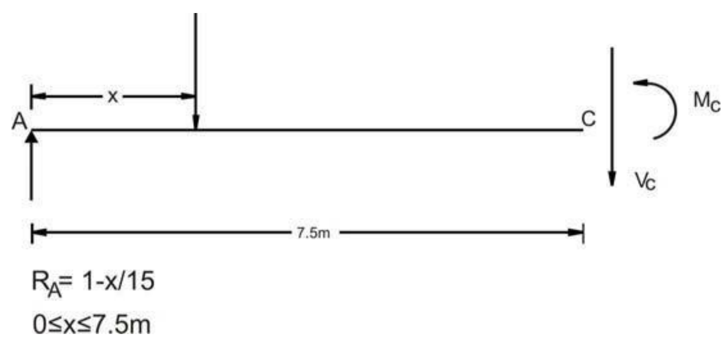


Figure 3: Free body diagram - a unit load before section

When the unit load is placed after point C then the moment equation for given Figure 4 can be given by

$\Sigma M_C = 0 : M_C - (1 - x/15) \times 7.5 = 0 \Rightarrow M_C = 7.5 - x/2$, where $7.5 < x \le 15.0$

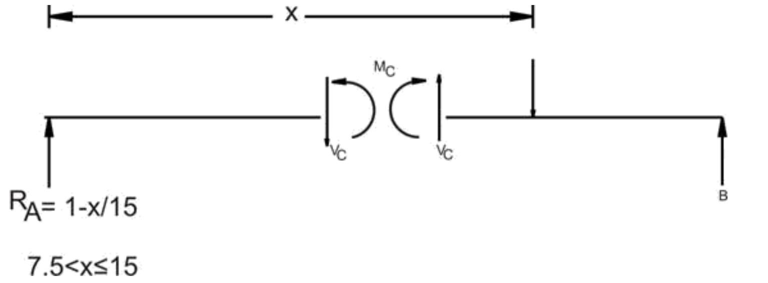


Figure 4: Free body diagram - a unit load before section

The equations are plotted in Figure 2

5. Construct the influence line for the moment at point C of the beam shown in Figure

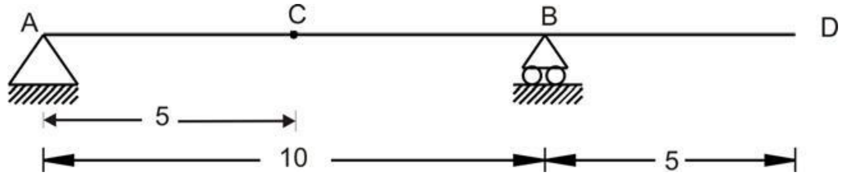


Figure: Overhang beam structure

Solution:

Tabulated values:

Place a unit load at different location between two supports and find the support reactions. Once the support reactions are computed, take a section at C and compute the moment. For example as shown in Figure 37.20, we place a unit load at 2.5 m from support A, then the support reaction at A will be 0.75 and support reaction B will be 0.25.

Figure 1: A unit load before section C

Taking section at C and computation of moment at C can be given by

$\Sigma M_C = 0 : - M_C + R_B \text{ x } 5.0 - = 0 \Rightarrow - M_C + 0.25 \text{ x } 5.0 = 0 \Rightarrow M_C = 1.25$

Similarly, compute the moment M_C for difference unit load position in the span.

The values of Mc are tabulated as follows.

x	M_C
0	0
2.5	1.25
5.0	2.5
7.5	1.25
10	0
12.5	-1.25
15.0	-2.5

Graphical representation of influence line for M_C is shown in Figure 2.

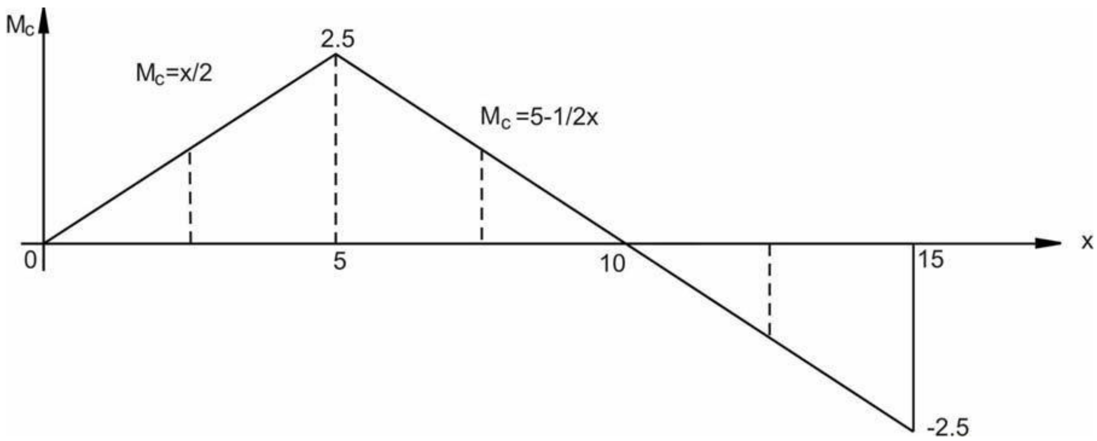


Figure 2: Influence line of moment at section C

Influence Line Equations:

There will be two influence line equations for the section before point C and after point C.

When a unit load is placed before point C then the moment equation for given Figure 3 can be given by

$\Sigma M_C = 0 : M_C + 1(5.0 - x) - (1 - x/10)x5.0 = 0 \Rightarrow M_C = x/2$, where $0 \le x \le 5.0$

Figure 3: A unit load before section C

When a unit load is placed after point C then the moment equation for given Figure 4 can be given by

$\Sigma M_C = 0 : M_C - (1 - x/10) \times 5.0 = 0 \Rightarrow M_C = 5 - x/2$, where $5 < x \le 15$

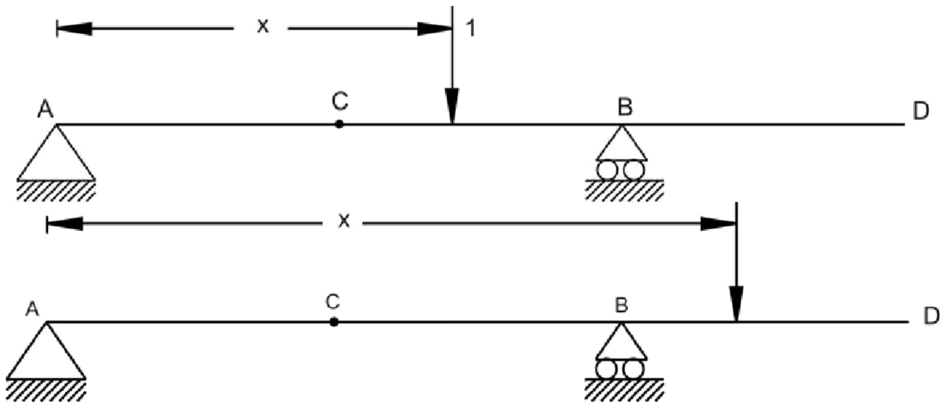


Figure 4: A unit load after section C

The equations are plotted in Figure 2.

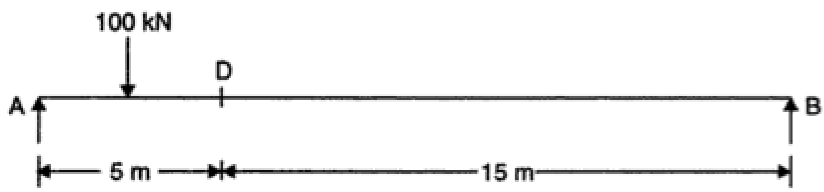
6. A Single rolling load of 100 kN moves on a girder of span 20m

(a) Construct the influence lines for

(i) Shear force and (ii) bending moment for a section 5m from the left support.

(b) Construct the influence lines for points at which the maximum shears and maximum bending moment develop. Determine these maximum values.

Solution:



a) To find maximum shear force and bending moment at 5m from the left support:

For the ILD for shear,

IL ordinate to the right of D = $\frac{l-x}{l} = \frac{20-5}{20} = 0.75$

IL ordinate to the left of D = $\frac{x}{l} = \frac{5}{20} = 0.25$

For the IL for bending moment, IL ordinate at D = $\frac{x(l-x)}{l} = \frac{5 \times 15}{20} = 3.75$ m.

i. **Maximum positive shear force**

By inspection of the ILD for shear force, it is evident that maximum positive shear force occurs when the load is placed just to the right of D

Maximum positive shear force = load x ordinate = 100 x 0.75 =75N

At D, SF_{max} + = 75 kN

UNIT-III
SLOPE DEFLECTION METHOD

INTRODUCTION

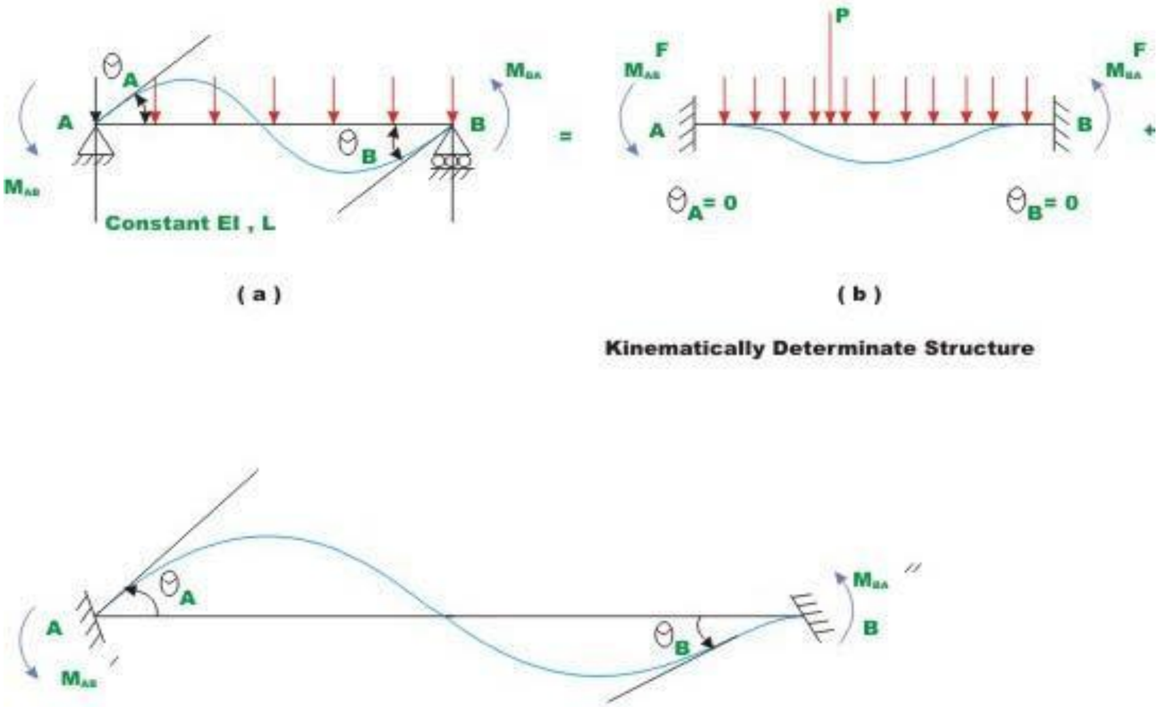
As pointed out earlier, there are two distinct methods of analysis for statically indeterminate structures depending on how equations of equilibrium, load displacement and compatibility conditions are satisfied: 1) force method of analysis and (2) displacement method of analysis. In the last module, force method of analysis was discussed. In this module, the displacement method of analysis will be discussed. In the force method of analysis, primary unknowns are forces and compatibility of displacements is written in terms of pre-selected redundant reactions and flexibility coefficients using force displacement relations. Solving these equations, the unknown redundant reactions are evaluated. The remaining reactions are obtained from equations of equilibrium.
As the name itself suggests, in the displacement method of analysis, the primary unknowns are displacements. Once the structural model is defined for the problem, the unknowns are automatically chosen unlike the force method. Hence this method is more suitable for computer implementation. In the displacement method of analysis, first equilibrium equations are satisfied. The equilibrium of forces is written by expressing the unknown joint displacements in terms of load by using load displacement relations. These equilibrium equations are solved for unknown joint displacements. In the next step, the unknown reactions are computed from compatibility equations using force displacement relations. In displacement method, three methods which are closely related to each other will be discussed.

1) Slope-Deflection Method
2) Moment Distribution Method

Degrees of freedom

In the displacement method of analysis, primary unknowns are joint displacements which are commonly referred to as the degrees of freedom of the structure. It is necessary to consider all the independent degrees of freedom while writing the equilibrium equations.These degrees of freedom are specified at supports, joints and at the free ends. For example, a propped cantilever beam (see Fig.14.01a) under the action of load P will undergo only rotation at B if axial deformation is neglected. In this case kinematic degree of freedom of the beam is only one i.e. θ_B as shown in the figure.

In Fig.14.01b, we have nodes at *A,B,C* and *D*. Under the action of lateral loads P_1, P_2 and P_3 , this continuous beam deform as shown in the figure. Here axial deformations are neglected. For this beam we have five degrees of freedom θ_A ,θ_B ,θ_C , θ_D and D as indicated in the figure. In Fig.14.02a, a symmetrical plane frame is loaded symmetrically. In this case we have only two degrees of freedomθ_B andθ_C . Now consider a frame as shown in Fig.14.02b. It has three degrees of freedom viz. θ_B ,θ_C and $_D$ as shown. Under the action of horizontal and vertical load, the frame will be displaced as shown in the figure. It is observed that nodes at *B* and *C* undergo rotation and also get displaced horizontally by an equal amount.



(a)

(b)

Kinematically Determinate Structure

(c)

Moment - Rotation relation

Fig.14.2 Derivation of slope - deflection equations

Hence in plane structures, each node can have at the most one linear displacement and one rotation. In this module first slope-deflection equations as applied to beams and rigid frames will be discussed.

Instructional Objectives

After reading this chapter the student will be able to
1. Calculate kinematic degrees of freedom of continuous beam.
2. Derive slope-deflection equations for the case beam with unyielding supports.
3. Differentiate between force method and displacement method of analyses.
4. State advantages of displacement method of analysis as compared to force method of analysis.
5. Analyse continuous beam using slope-deflection method.

Introduction

In this lesson the slope-deflection equations are derived for the case of a beam with unyielding supports .In this method, the unknown slopes and deflections at nodes are related to the applied loading on the structure. As introduced earlier, the slope-deflection method can be used to analyze statically determinate and indeterminate beams and frames. In this method it is assumed that all deformations are due to bending only. In other words deformations due to axial forces are neglected. As discussed earlier in the force method of analysis compatibility equations are written in terms of unknown reactions. It must be noted that all the unknown reactions appear in each of the compatibility equations making it difficult to solve resulting equations. The slope-deflection equations are not that lengthy in comparison.
The slope-deflection method was originally developed by Heinrich Manderla and Otto Mohr for computing secondary stresses in trusses. The method as used today was presented by G.A.Maney in 1915 for analyzing rigid jointed structures.

Slope-Deflection Equations

Consider a typical span of a continuous beam AB as shown in Fig.14.1.The beam has constant flexural rigidity EI and is subjected to uniformly distributed loading and concentrated loads as shown in the figure. The beam is kinematically indeterminate to second degree. In this lesson, the slope-deflection equations are derived for the simplest case i.e. for the case of continuous beams with unyielding supports. In the next lesson, the support settlements are included in the slope-deflection equations.

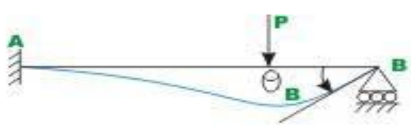


Fig. 14.01

For this problem, it is required to derive relation between the joint end moments M_{AB} and M_{BA} in terms of joint rotations θ_A and θ_B and loads acting on the beam .Two subscripts are used to denote end moments. For example, end moments M_{AB} denote moment acting at joint *A* of the member *AB*. Rotations of the tangent to the elastic curve are denoted by one subscript. Thus, θ_A denotes the rotation of the tangent to the elastic curve at A. The following sign conventions are used in the slope-deflection equations (1) Moments acting at the ends of the member in counterclockwise direction are taken to be positive. (2) The rotation of the tangent to the elastic curve is taken to be positive when the tangent to the elastic curve has rotated in the counterclockwise direction from its original direction. The slope-deflection equations are derived by superimposing the end moments developed due to (1) applied loads (2) rotation θ_A (3) rotationθ_B . This is shown in Fig.14.2 (a)-(c). In Fig. 14.2(b) a kinematically determinate structure is obtained. This condition is obtained by modifying the support conditions to fixed so that the unknown joint rotations become zero. The structure shown in Fig.14.2 (b) is known as kinematically determinate structure or restrained structure. For this case, the end moments are denoted by M_{AB}^{F} and M_{BA}^{F} . The fixed end moments are evaluated by force–method of analysis as discussed in the previous module. For example for fixed- fixed beam subjected to uniformly distributed load, the fixed-end moments are shown in Fig.14.3.

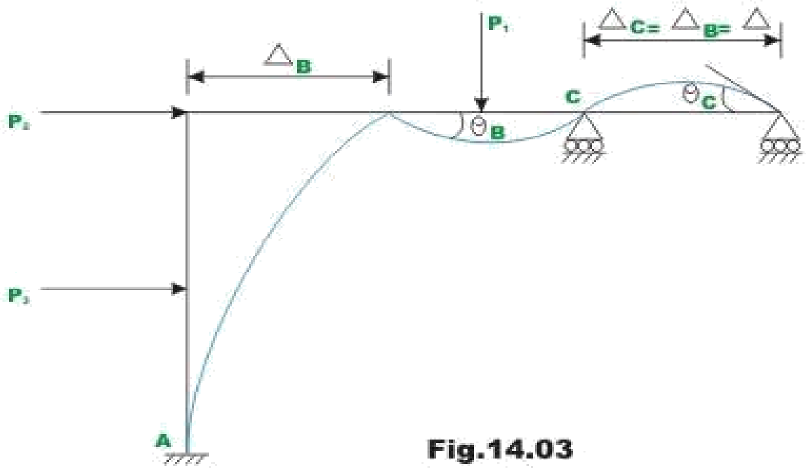


Fig.14.03

The fixed end moments are required for various load cases. For ease of calculations, fixed end forces for various load cases are given at the end of this lesson. In the actual structure end A rotates by θ_A and end B rotates by θ_B . Now it is required to derive a relation relating θ_A and θ_B with the end moments M'_{AB} and M'_{BA} . Towards this end, now consider a simply supported beam acted by moment M_{AB}' at A as shown in Fig. 14.4. The end moment M_{AB}' deflects the beam as shown in the figure. The rotations $\theta_A'$and $\theta_B'$are calculated from moment-area theorem.

$$\theta_A' = \frac{M'_{AB} L}{3EI} \qquad (14.1a)$$

$$\theta_B' = -\frac{M'_{AB} L}{6EI} \qquad (14.1b)$$

Now a similar relation may be derived if only $M_{BA}'$is acting at end B (see Fig. 14.4).

$$\theta_B'' = \frac{M'_{BA} L}{3EI} \quad \text{and} \qquad (14.2a)$$

$$\theta_A'' = -\frac{M'_{BA} L}{6EI} \qquad (14.2b)$$

Now combining these two relations, we could relate end moments acting at A and B to rotations produced at A and B as (see Fig. 14.2c)

$$\theta_A = \frac{M'_{AB} L}{3EI} - \frac{M'_{BA} L}{6EI} \qquad (14.3a)$$

$$\theta_B = \frac{M'_{BA} L}{3EI} - \frac{M'_{AB} L}{6EI} \qquad (14.3b)$$

Solving for M'_{AB} and M'_{BA} in terms of θ_A and θ_B ,

$$M'_{AB} = \frac{2EI}{L}(2\theta_A + \theta_B) \qquad (14.4)$$

$$M'_{BA} = \frac{2EI}{L}(2\theta_B + \theta_A) \qquad (14.5)$$

Now writing the equilibrium equation for joint moment at A (see Fig. 14.2).

$$M_{AB} = M_{AB}^F + M'_{AB} \qquad (14.6a)$$

Similarly writing equilibrium equation for joint B

$$M_{BA} = M_{BA}^F + M'_{BA} \qquad (14.6b)$$

Substituting the value of M'_{AB} from equation (14.4) in equation (14.6a) one obtains,

$$M_{AB} = M_{AB}^F + \frac{2EI}{L}(2\theta_A + \theta_B) \qquad (14.7a)$$

Similarly substituting M'_{BA} from equation (14.6b) in equation (14.6b) one obtains,

$$M_{BA} = M_{BA}^F + \frac{2EI}{L}(2\theta_B + \theta_A) \qquad (14.7b)$$

Sometimes one end is referred to as near end and the other end as the far end. In that case, the above equation may be stated as the internal moment at the near end of the span is equal to the fixed end moment at the near end due to external loads plus $\frac{2EI}{L}$ times the sum of twice the slope at the near end and the slope at the far end. The above two equations (14.7a) and (14.7b) simply referred to as slope–deflection equations. The slope-deflection equation is nothing but a load displacement relationship.

Application of Slope-Deflection Equations to Statically Indeterminate Beams.

The procedure is the same whether it is applied to beams or frames. It may be summarized as follows:

1. Identify all kinematic degrees of freedom for the given problem. This can be done by drawing the deflection shape of the structure. All degrees of freedom are treated as unknowns in slope-deflection method.
2. Determine the fixed end moments at each end of the span to applied load. The table given at the end of this lesson may be used for this purpose.
3. Express all internal end moments in terms of fixed end moments and near end, and far end joint rotations by slope-deflection equations.
4. Write down one equilibrium equation for each unknown joint rotation. For example, at a support in a continuous beam, the sum of all moments corresponding to an unknown joint rotation at that support must be zero. Write down as many equilibrium equations as there are unknown joint rotations.
5. Solve the above set of equilibrium equations for joint rotations.
6. Now substituting these joint rotations in the slope-deflection equations evaluate the end moments.
7. Determine all rotations.

Example

A continuous beam *ABC* is carrying uniformly distributed load of 2 kN/m in addition to a concentrated load of 20 kN as shown in Fig.14.5a. Draw bending moment and shear force diagrams. Assume *EI* to be constant.

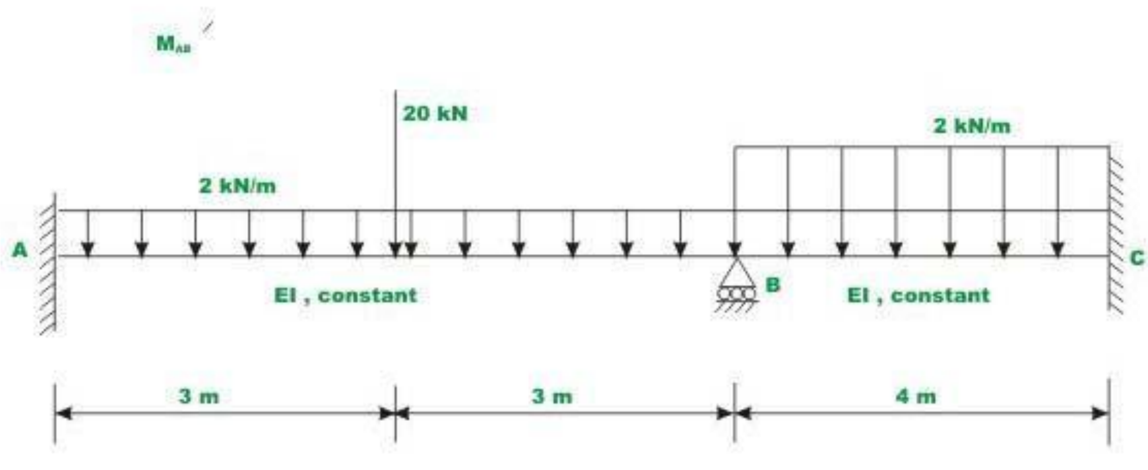


Fig. 14.5(a) Example 14.1

(a). Degrees of freedom

It is observed that the continuous beam is kinematically indeterminate to first degree as only one joint rotation θ_B is unknown. The deflected shape /elastic curve of the beam is drawn in Fig.14.5b in order to identify degrees of freedom. By fixing the support or restraining the support B against rotation, the fixed-fixed beams area obtained as shown in Fig.14.5c.

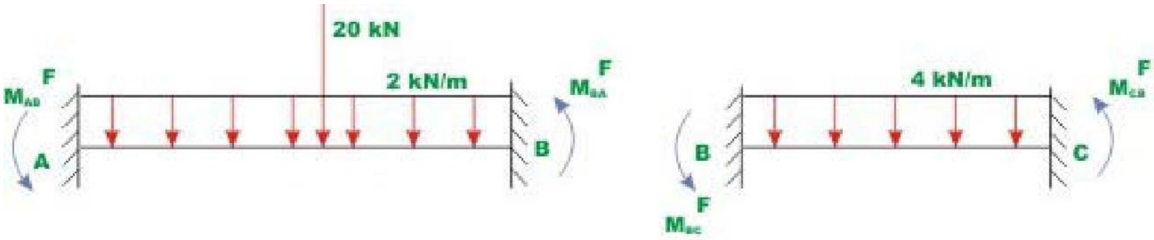


Fig. 14.5 (c) Restrained Structure.

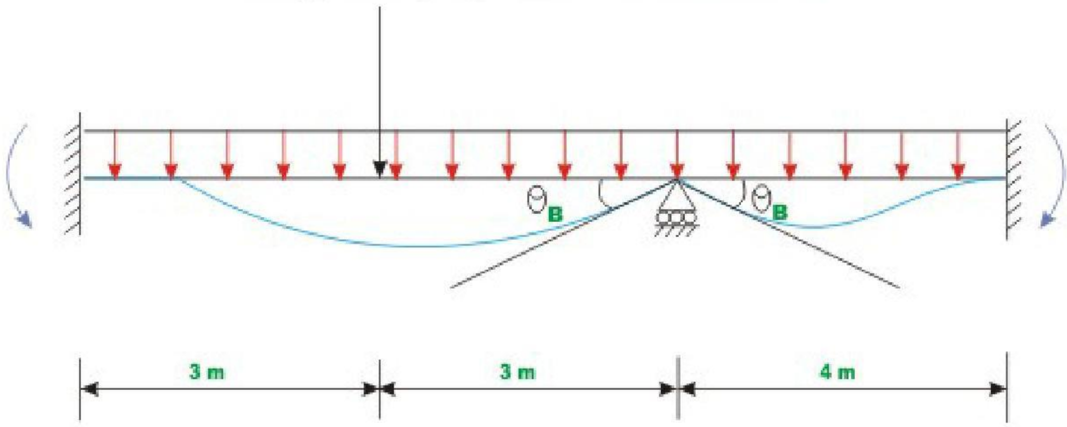


Fig. 14.5 (b) Elastic curve of the beam with unknown displacement component θ_B

(b). Fixed end moments M_{AB}^F , M_{BA}^F , M_{BC}^F and M_{CB}^F are calculated referring to the Fig. 14. and following the sign conventions that counterclockwise moments are positive.

$$M_{AB}^F = \frac{2\times 6^2}{12} + \frac{20\times 3\times 3^2}{6^2} = 21 \text{ kN.m}$$

$$M_{BA}^F = -21 \text{ kN.m}$$

$$M_{BC}^F = \frac{4\times 4^2}{12} = 5.33 \text{ kN.m}$$

$$M_{CB}^F = -5.33 \text{ kN.m} \qquad (1)$$

(c) Slope-deflection equations

Since ends A and C are fixed, the rotation at the fixed supports is zero, $\theta_A = \theta_C = 0$. Only one non-zero rotation is to be evaluated for this problem. Now, write slope-deflection equations for span AB and BC.

$$M_{AB} = M_{AB}^F + \frac{2EI}{l}(2\theta_A + \theta_B)$$

$$M_{AB} = 21 + \frac{2EI}{6}\theta_B \qquad (2)$$

$$M_{BA} = -21 + \frac{2EI}{l}(2\theta_B + \theta_A)$$

$$M_{BA} = -21 + \frac{4EI}{6}\theta_B \qquad (3)$$

$$M_{BC} = 5.33 + EI\theta_B \qquad (4)$$

$$M_{CB} = -5.33 + 0.5EI\theta_B \qquad (5)$$

(d) Equilibrium equations

In the above four equations (2-5), the member end moments are expressed in terms of unknown rotation θ_B. Now, the required equation to solve for the rotation θ_B is the moment equilibrium equation at support B. The free body diagram of support B along with the support moments acting on it is shown in Fig. 14.5d. For, moment equilibrium at support B, one must have,

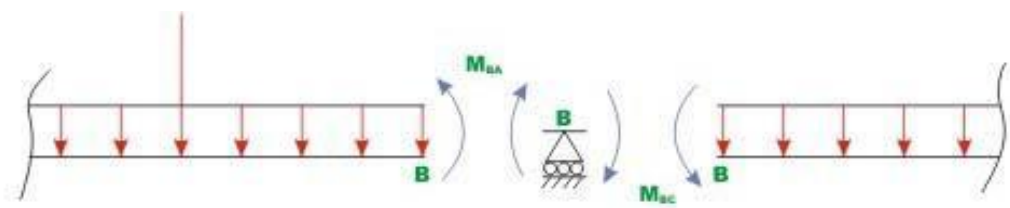


Fig. 14.5 d Free- body diagram of the joint B

$$\sum M_B = 0 \qquad M_{BA} + M_{BC} = 0 \qquad (6)$$

Substituting the values of M_{BA} and M_{BC} in the above equilibrium equation,

$$-21 + \frac{4EI}{6}\theta_B + 5.33 + EI\theta_B = 0$$

$$\Rightarrow 1.667\theta_B EI = 15.667$$

$$\theta_B = \frac{9.398}{EI} \cong \frac{9.40}{EI} \qquad (7)$$

(e) End moments

After evaluating θ_B, substitute it in equations (2-5) to evaluate beam end moments. Thus,

$$M_{AB} = 21 + \frac{EI}{3}\theta_B$$

$$M_{AB} = 21 + \frac{EI}{3} \times \frac{9.398}{EI} = 24.133 \text{kN.m}$$

$$M_{BA} = -21 + \frac{EI}{3}(2\theta_B)$$

$$M_{BA} = -21 + \frac{EI}{3} \times \frac{2 \times 9.4}{EI} = -14.733 \text{kN.m}$$

$$M_{BC} = 5.333 + \frac{9.4}{EI} EI = 14.733 \text{kN.m}$$

$$M_{CB} = -5.333 + \frac{9.4}{EI} \times \frac{EI}{2} = -0.63 \text{ kN.m} \qquad (8)$$

(f) Reactions

Now, reactions at supports are evaluated using equilibrium equations (vide Fig. 14.5e)

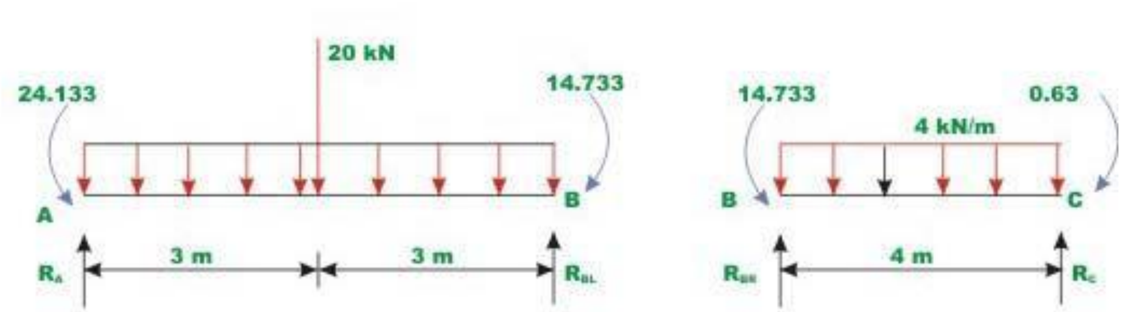


Fig. 14.5 (e) Free - body diagram of two members

$$R_A \times 6 + 14.733 - 20 \times 3 - 2 \times 6 \times 3 - 24.133 = 0$$

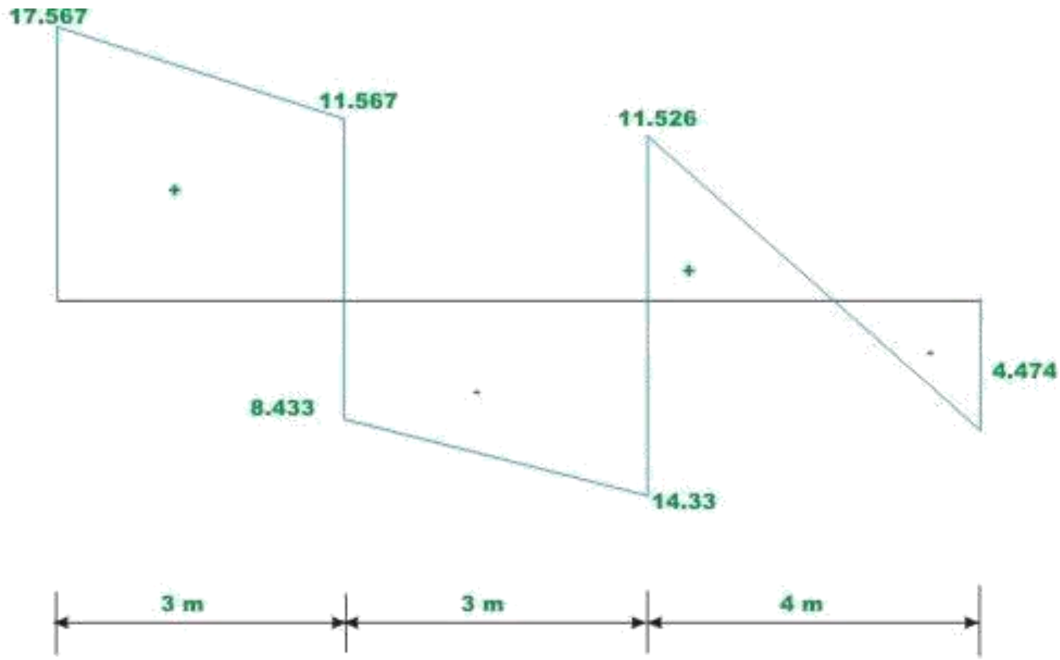
$$R_A = 17.567 \text{ kN}(\uparrow)$$

$$R_{BL} = 16 - 1.567 = 14.433 \text{ kN}(\uparrow)$$

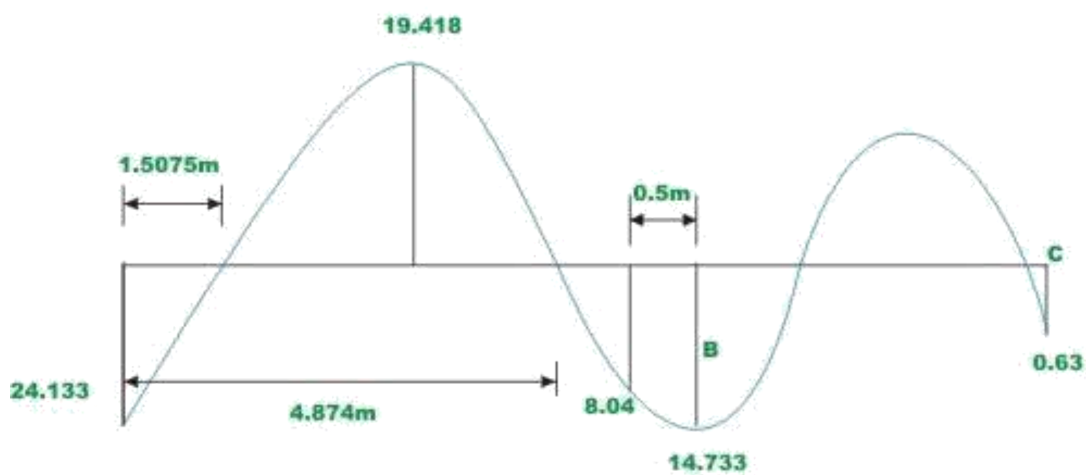
$$R_{BR} = 8 + \frac{14.733 - 0.63}{4} = 11.526 \text{ kN}(\uparrow)$$

$$R_C = 8 + 3.526 = 4.47 \text{ kN}(\uparrow) \qquad (9)$$

The shear force and bending moment diagrams are shown in Fig. 14.5f.



Shear force diagram



Bending Moment diagram

Fig. 14.5 f. Shear force and bending moment diagram of continuous beam ABC

Example

Draw shear force and bending moment diagram for the continuous beam *ABCD* loaded as shown in Fig.14.6a.The relative stiffness of each span of the beam is also shown in the figure.

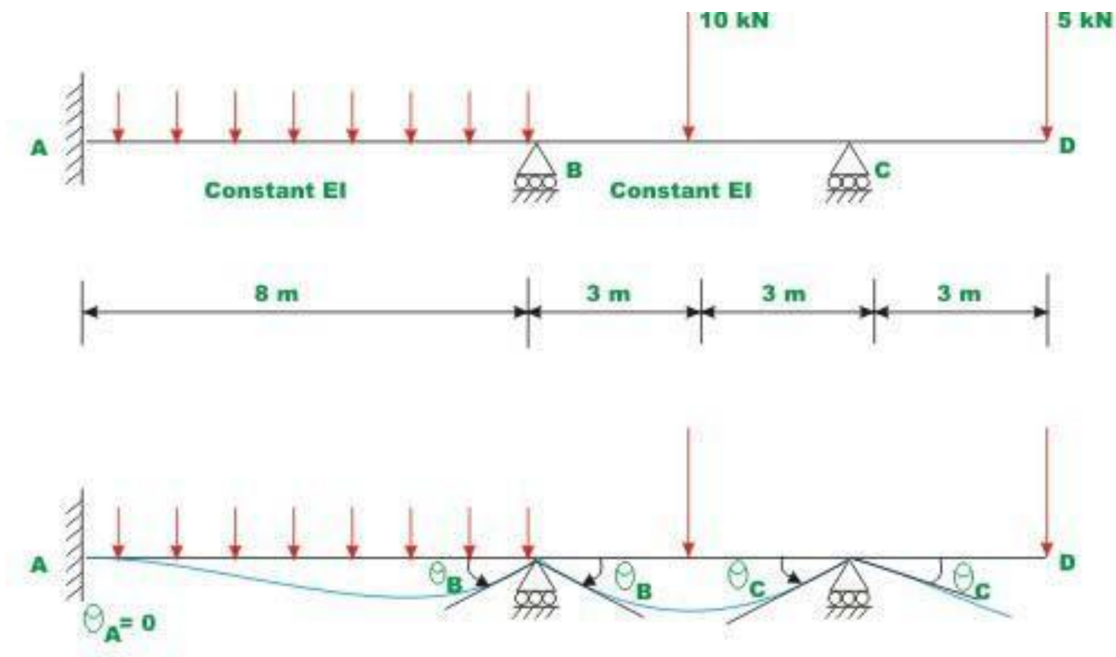


Fig. 14.6a Continuous beam of Example 14.2

For the cantilever beam portion *CD*, no slope-deflection equation need to be written as there is no internal moment at end *D*. First, fixing the supports at *B* and *C*, calculate the fixed end moments for span *AB* and *BC*. Thus,

$$M_{AB}^{F} = \frac{3 \times 8^2}{12} = 16 \text{ kN.m}$$

$$M_{BA}^{F} = -16 \text{ kN.m}$$

$$M_{BC}^{F} = \frac{10 \times 3 \times 3^2}{6^2} = 7.5 \text{ kN.m}$$

$$M_{CB}^{F} = -7.5 \text{ kN.m} \qquad (1)$$

In the next step write slope-deflection equation. There are two equations for each span of the continuous beam.

$$M_{AB} = 16 + \frac{2EI}{8}(\theta_B) = 16 + 0.25\theta_B EI$$

$$M_{BA} = -16 + 0.5\theta_B EI$$

$$M_{BC} = 7.5 + \frac{2 \times 2EI}{6}(2\theta_B + \theta_C) = 7.5 + 1.334EI\theta_B + 0.667EI\theta_C$$

$$M_{CB} = -7.5 + 1.334EI\theta_C + 0.667EI\theta_B \qquad (2)$$

Equilibrium equations

The free body diagram of members AB , BC and joints B and C are shown in Fig.14.6b.One could write one equilibrium equation for each joint B and C.

Fig. 14.6 b Free - body diagrams of joints B and C along with members

Support B,

$\sum M_B = 0 \qquad M_{BA} + M_{BC} = 0$ (3)

$\sum M_C = 0 \qquad M_{CB} + M_{CD} = 0$ (4)

We know that $M_{CD} = 15$ kN.m (5)

$\Rightarrow M_{CB} = -15$ kN.m (6)

Substituting the values of M_{CB} and M_{CD} in the above equations for M_{AB} , M_{BA} , M_{BC} and M_{CB} we get,

$$\theta_B = \frac{24.5}{3.001} = 8.164$$

$\theta_C = 9.704$ (7)

Substituting θ_B ,θ_C in the slope-deflection equations, we get

$$M_{AB} = 16 + 0.25\,EI\,\theta_B = 16 + 0.25EI \times \frac{8.164}{EI} = 18.04 \text{ kN.m}$$

$$M_{BA} = -16 + 0.5EI\,\theta_B = -16 + 0.5EI \times \frac{8.164}{EI} = -11.918 \text{ kN.m}$$

$$M_{BC} = 7.5 + 1.334EI \times \frac{8.164}{EI} + 0.667EI\left(\frac{9.704}{EI}\right) = 11.918 \text{ kN.m}$$

$$M_{CB} = -7.5 + 0.667EI \times \frac{8.164}{EI} + 1.334EI\left(-\frac{9.704}{EI}\right) = -15 \text{ kN.m} \qquad (8)$$

Reactions are obtained from equilibrium equations (ref. Fig. 14.6c)

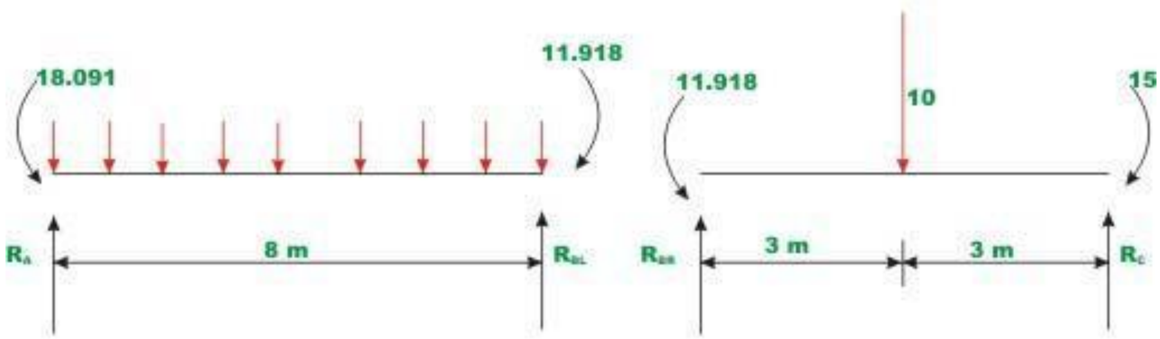


Fig. 14.6 c Computation of reactions

$$R_A \times 8 - 18.041 - 3 \times 8 \times 4 + 11.918 = 0$$

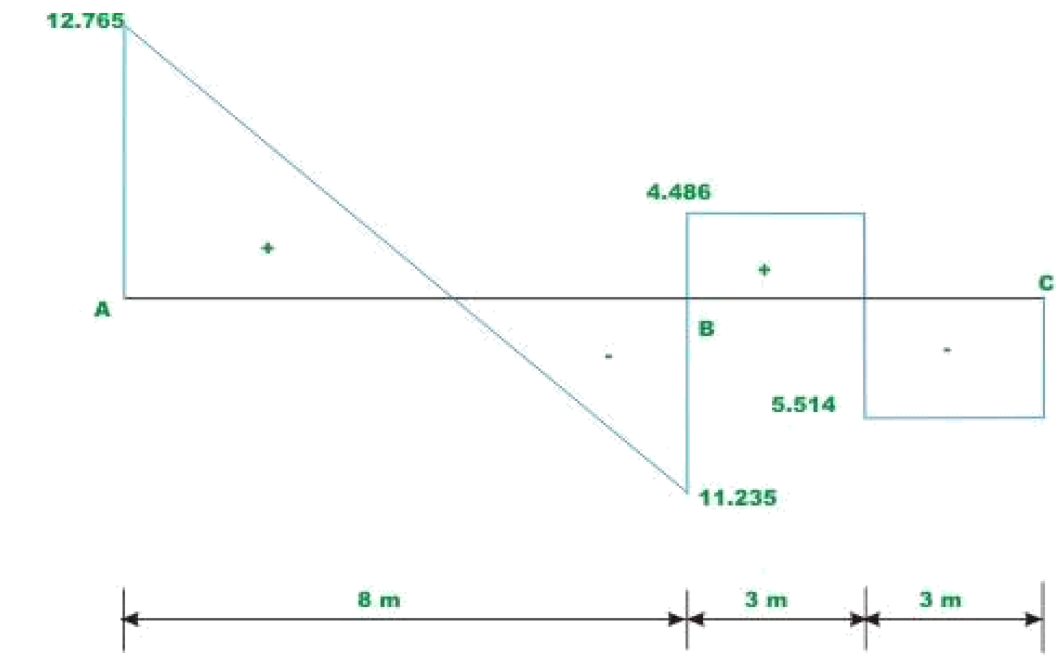
$$R_A = 12.765 \text{ kN}$$

$$R_{BR} = 5 - 0.514kN = 4.486 \text{ kN}$$

$$R_{BL} = 11.235 \text{ kN}$$

$$R_C = 5 + 0.514kN = 5.514 \text{ kN}$$

The shear force and bending moment diagrams are shown in Fig. 14.6d.



Shear force diagram

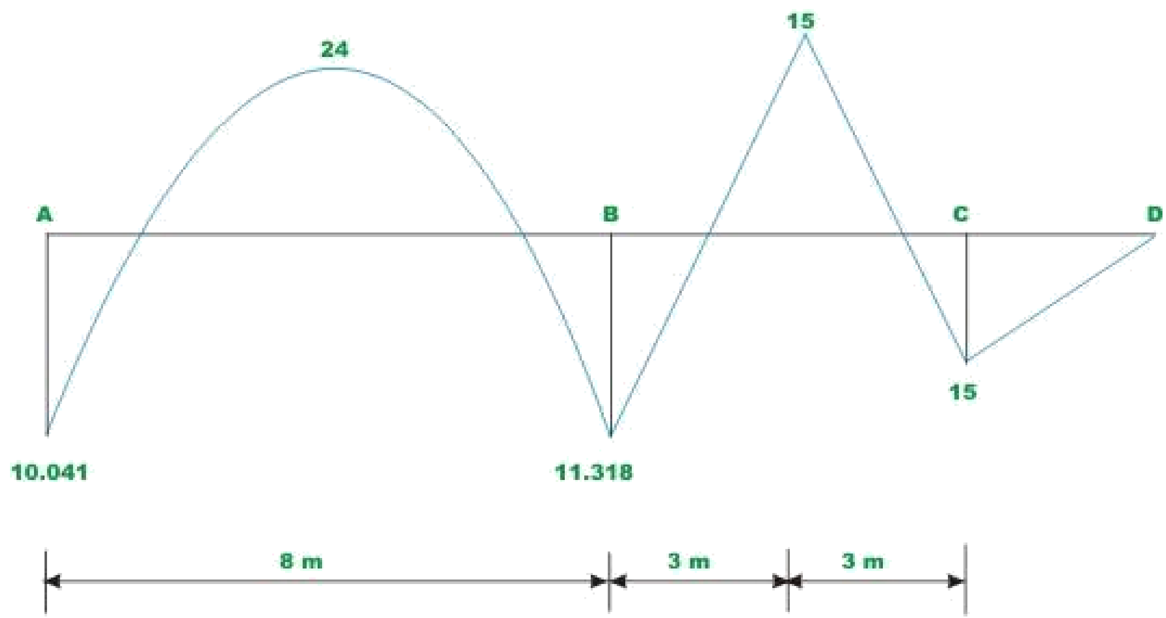


Fig. 14.6 (d) Shear force and bending moment diagram

For ease of calculations, fixed end forces for various load cases are given in Fig. 14.7.

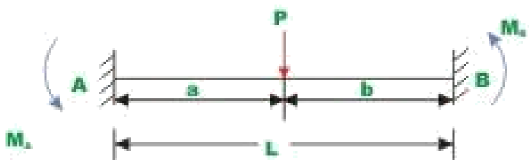
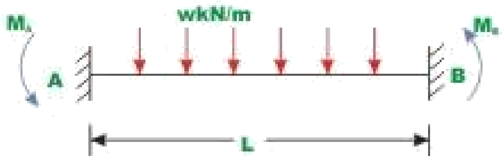
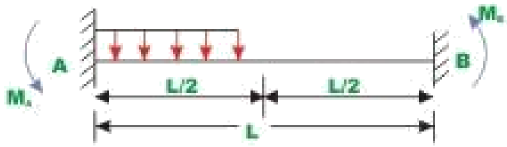
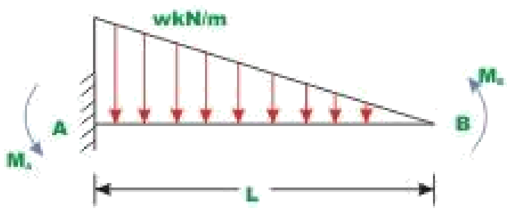
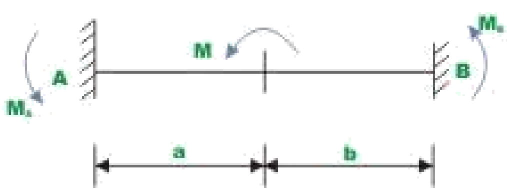
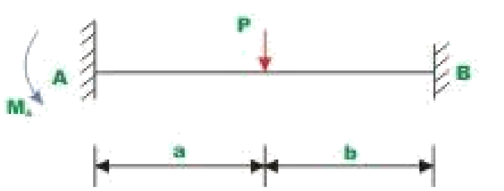
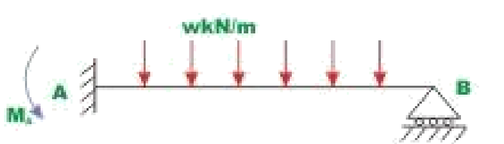
Load case	M_A	M_B
	(+ve Counter clockwise)	
	$M_A = \frac{Pab^2}{L^2}$	$M_B = -\frac{Pab^2}{L^2}$
	$M_A = \frac{wL^2}{12}$	$M_B = -\frac{wL^2}{12}$
	$M_A = \frac{11wL^2}{192}$	$M_B = -\frac{5wL^2}{192}$
	$M_A = \frac{wL^2}{20}$	$M_B = -\frac{wL^2}{30}$
	$M_A = \frac{Mb}{L^2}(2a - b)$	$M = \frac{Ma}{L^2}(2b - a)$
	$M_A = \frac{P}{L^2}\left(b^2a + \frac{a^2b}{2}\right)$	$M_B = 0$
	$M_A = \frac{wL^2}{8}$	$M_B = 0$

Fig. 14.7 Table of fixed end moments

Introduction

In this lesson, slope deflection equations are applied to solve the statically indeterminate frames without side sway. In frames axial deformations are much smaller than the bending deformations and are neglected in the analysis. With this assumption the frames shown in Fig 16.1 will not side sway. i.e. the frames will not be displaced to the right or left. The frames shown in Fig 16.1(a) and Fig 16.1(b) are properly restrained against side sway. For example in Fig 16.1(a) the joint can't move to the right or left without support A also moving .This is true also for joint D .Frames shown in Fig 16.1 (c) and (d) are not restrained against side sway. However the frames are symmetrical in geometry and in loading and hence these will not side sway. In general, frames do not side sway if

1) They are restrained against side sway.
2) The frame geometry and loading is symmetrical

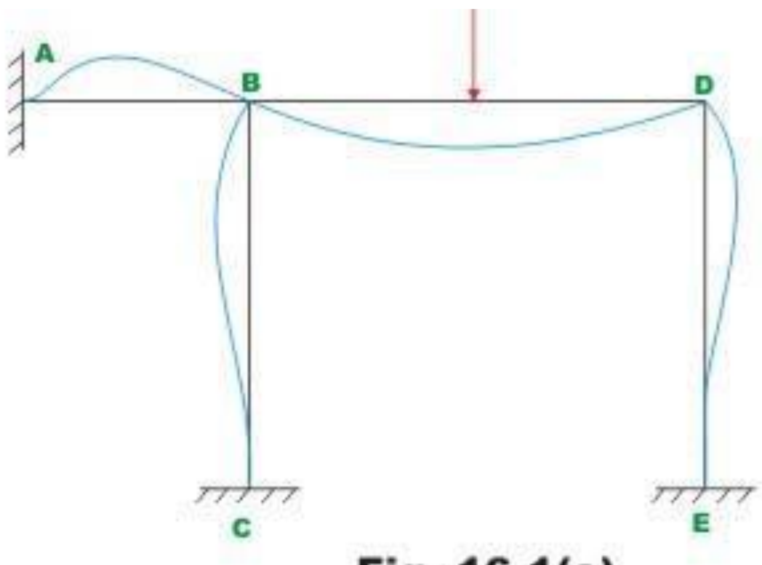


Fig- 16.1(a)

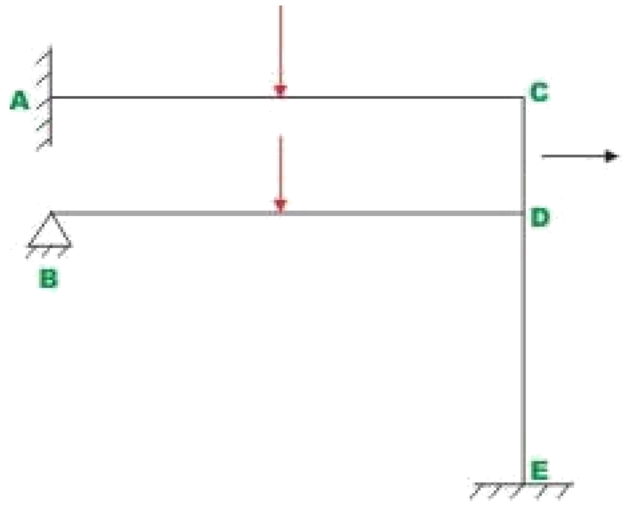


Fig- 16.1(b)

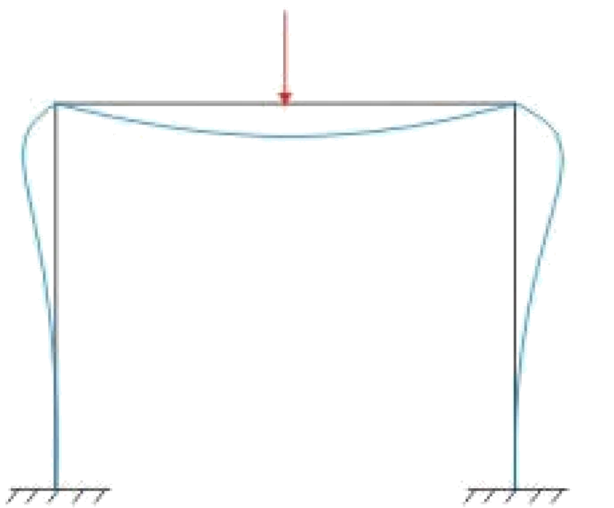


Fig- 16.1(c)

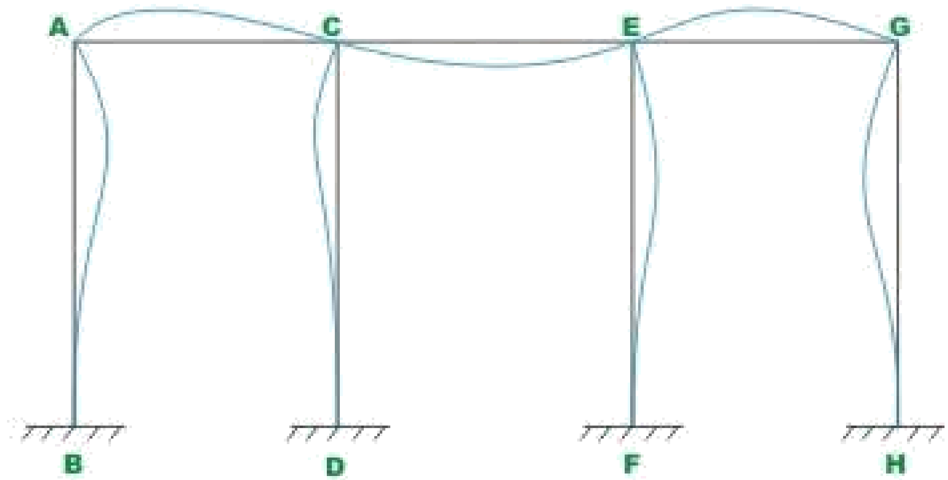


Fig- 16.1(d)

For the frames shown in Fig 16.1, the angle ψ in slope-deflection equation is zero. Hence the analysis of such rigid frames by slope deflection equation essentially follows the same steps as that of continuous beams without support settlements. However, there is a small difference. In the case of continuous beam, at a joint only two members meet. Whereas in the case of rigid frames two or more than two members meet at a joint. At joint C in the frame shown in Fig 16.1(d) three members meet. Now consider the free body diagram of joint C as shown in fig 16.2 .The equilibrium equation at joint C is

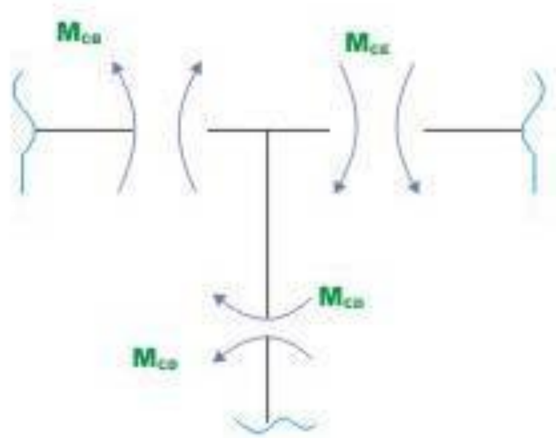


Fig- 16.2

$$\sum M_C = 0 \Rightarrow \quad M_{CB} + M_{CE} + M_{CD} = 0$$

At each joint there is only one unknown as all the ends of members meeting at a joint rotate by the same amount. One would write as many equilibrium equations as the no of unknowns, and solving these equations joint rotations are evaluated. Substituting joint rotations in the slope–deflection equations member end moments are calculated. The whole procedure is illustrated by few examples. Frames undergoing sidesway will be considered in next lesson.

Example

Analyse the rigid frame shown in Fig 16.3 (a). Assume EI to be constant for all the members. Draw bending moment diagram and also sketch the elastic curve.

Solution

In this problem only one rotation needs to be determined i. e. θ_B . Thus the required equations to evaluate θ_B is obtained by considering the equilibrium of joint B . The moment in the cantilever portion is known. Hence this moment is applied on frame as shown in Fig 16.3 (b). Now, calculate the fixed-end moments by fixing the support B (vide Fig 16.3 c). Thus

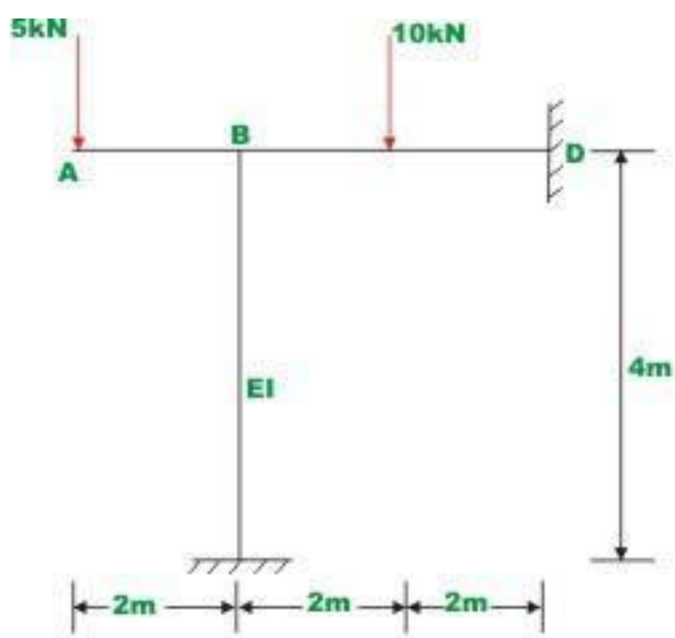


Fig- 16.3 a Example 16.1

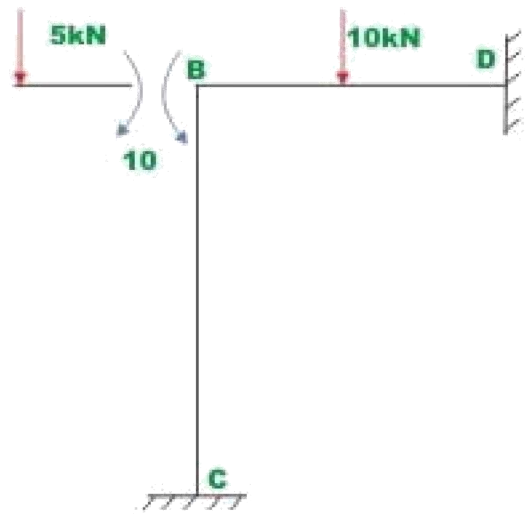


Fig- 16.3 b Moment at joint B due to overhang

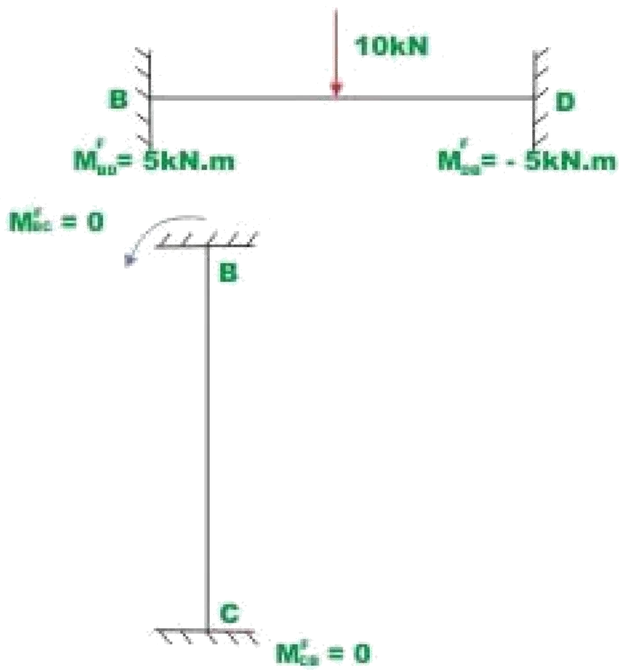


Fig- 16.3 © Kinematically restrained structure

$$M_{BD}^{F} = +5 \text{ kNm}$$

$$M_{DB}^{F} = -5 \text{ kNm}$$

$$M_{BC}^{F} = 0 \text{ kNm}$$

$$M_{BC}^{F} = 0 \text{ kNm}$$

For writing slope–deflection equations two spans must be considered, BC and BD. Since supports C and D are fixed $\theta_C = \theta_D = 0$. Also the frame is restrained against sidesway.

$$M_{BD} = 5 + \frac{2EI}{4}\left[2\theta_B\right] = 5 + EI\theta_B$$

$$M_{DB} = 5 + \frac{2EI}{4}\left[\theta_B\right] = -5 + 0.5EI\theta_B$$

$$M_{BC} = EI\theta_B$$

$$M_{CB} = 0.5EI\theta_B \qquad (2)$$

Now consider the joint equilibrium of support B, (see Fig 16.3 d)

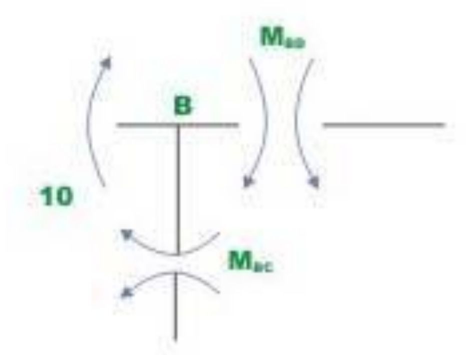


Fig- 16.3 (d) Free - body diagram of joint B

$$\sum M_B = 0 \Rightarrow \quad M_{BD} + M_{BC} - 10 = 0 \qquad (3)$$

Substituting the value of M_{BD} and M_{BC} and from equation (2) in the above equation

$$5 + EI\theta_B + EI\theta_B - 10 = 0$$

$$\theta_B = \frac{2.5}{EI} \qquad (4)$$

Substituting the values of θ_B in equation (2), the beam end moments are calculated

$$M_{BD} = +7.5 \text{ kN} \cdot \text{m}$$

$$M_{DB} = -3.75 \text{ kN} \cdot \text{m}$$

$$M_{BC} = +2.5 \text{ kN} \cdot \text{m}$$

$$M_{CB} = +1.25 \text{ kN} \cdot \text{m} \qquad (5)$$

The reactions are evaluated from static equations of equilibrium. The free body diagram of each member of the frame with external load and end moments are shown in Fig 16.3 (e)

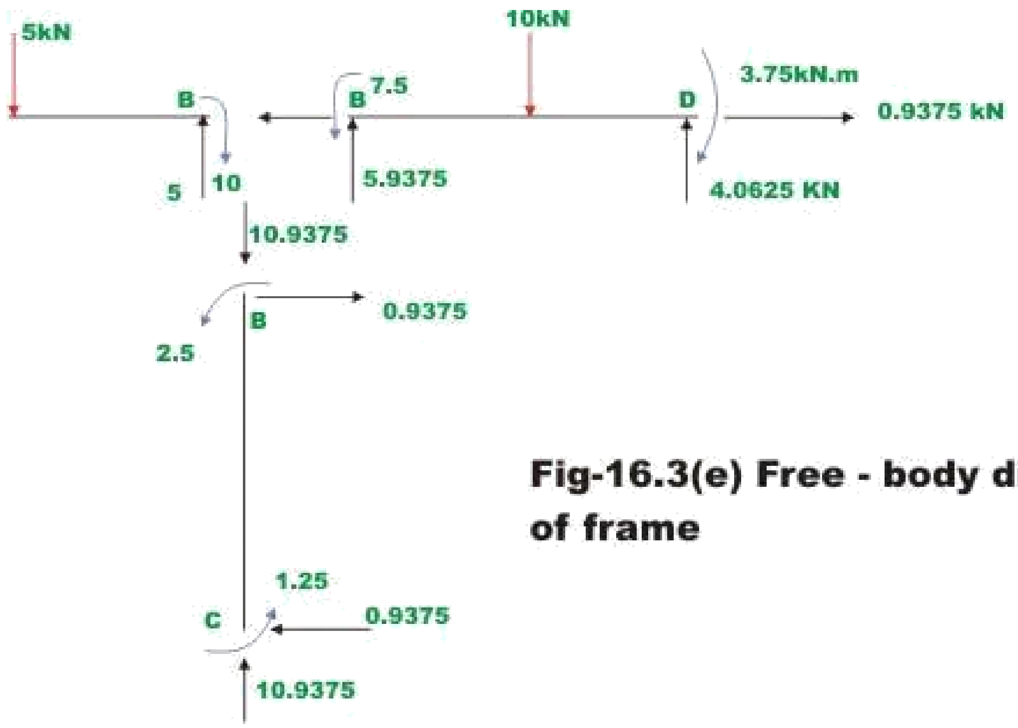


Fig-16.3(e) Free - body diagram of frame

$$R_{Cy} = 10.9375 \text{ kN}(\uparrow)$$

$$R_{Cx} = -0.9375 \text{ kN}(\leftarrow)$$

$$R_{Dy} = 4.0625 \text{ kN}(\uparrow)$$

$$R_{Dx} = 0.9375 \text{ kN}(\rightarrow) \quad (6)$$

Bending moment diagram is shown in Fig 16.3(f)

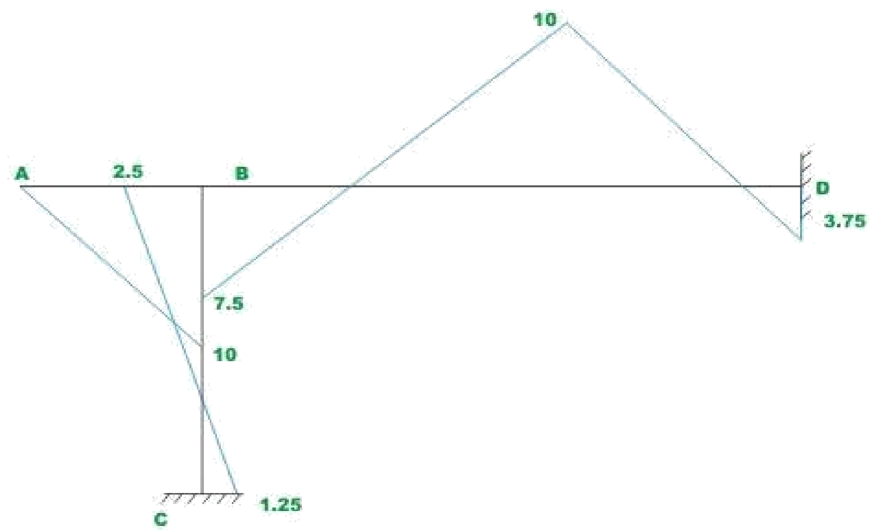


Fig-16.3(f) Bending moment diagram plotted on compression side

The vertical hatching is use to represent the bending moment diagram for the horizontal member (beams) and horizontal hatching is used for bending moment diagram for the vertical members.
The qualitative elastic curve is shown in Fig 16.3 (g).

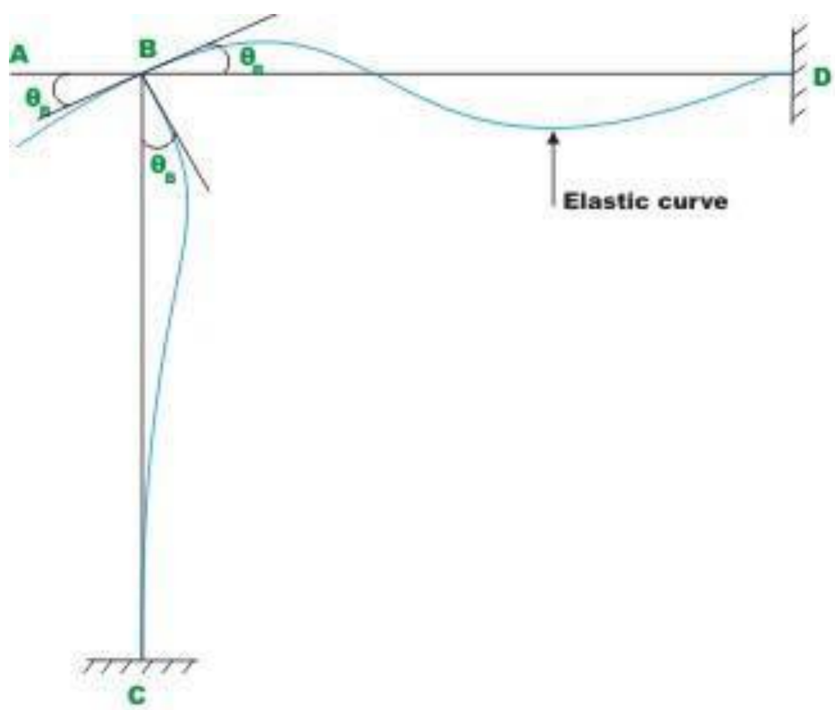


Fig-16.3(g) Elastic curve

Example

Compute reactions and beam end moments for the rigid frame shown in Fig 16.4 (a). Draw bending moment and shear force diagram for the frame and also sketch qualitative elastic curve.

Solution

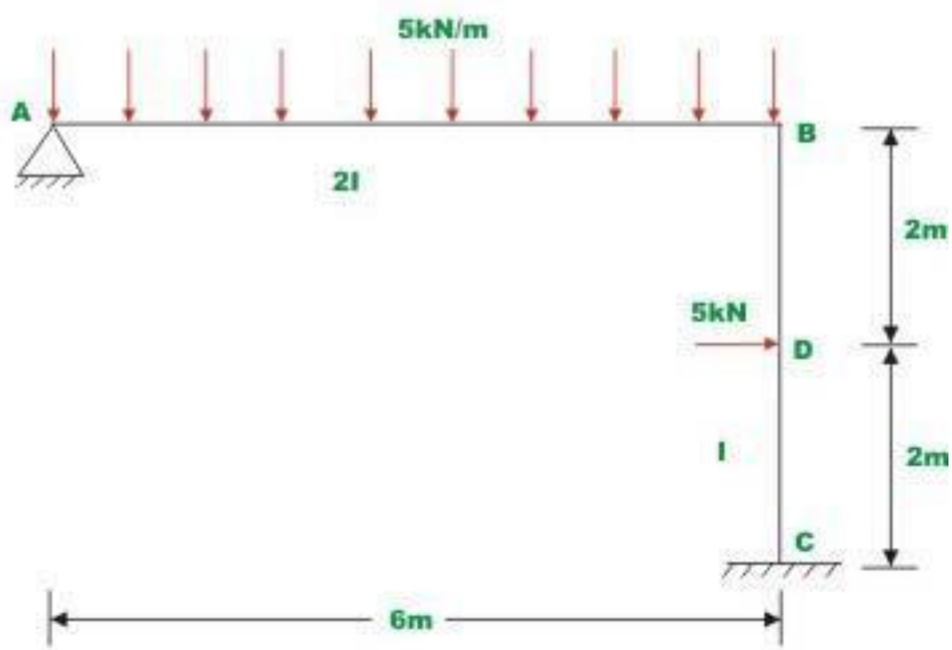


Fig-16.4(a) Example 16.2

In this frame rotations θ_A and θ_B are evaluated by considering the equilibrium of joint A and B . The given frame is kinematically indeterminate to second degree. Evaluate fixed end moments. This is done by considering the kinematically determinate structure. (Fig 16.4 b)

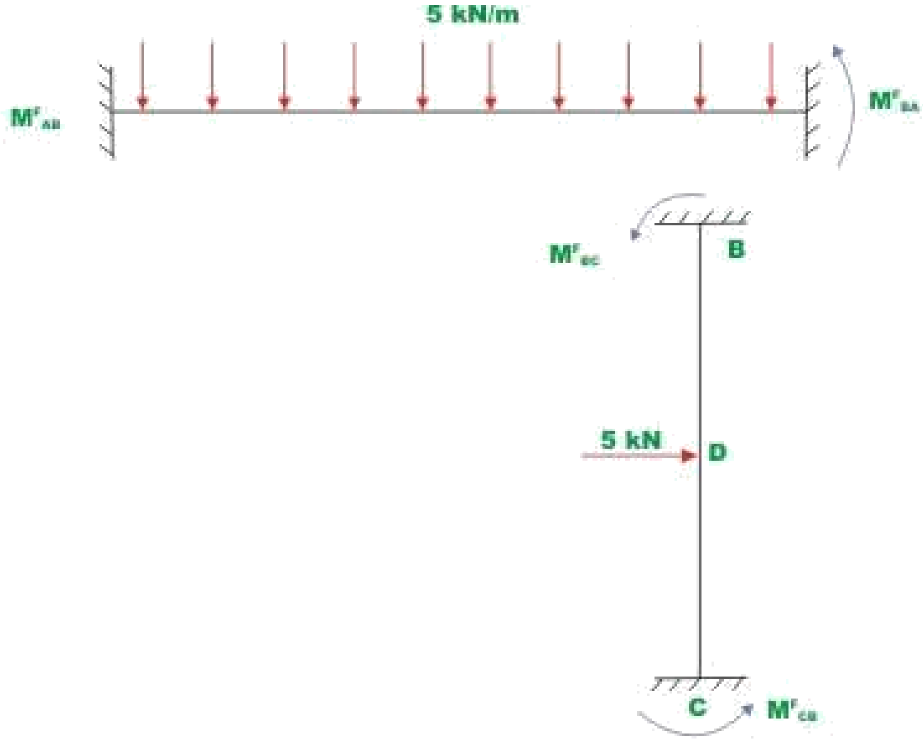


Fig-16.4(b) Kinematically restrained structure

$$M_{DB}^{F} = \frac{5\times 6^2}{12} = 15 \text{ kN.m}$$

$$M_{BA}^{F} = \frac{-5\times 6^2}{12} = -15 \text{ kN.m}$$

$$M_{BC}^{F} = \frac{5\times 2\times 2^2}{4^2} = 2.5 \text{ kN.m}$$

$$M_{CD}^{F} = \frac{-5\times 2\times 2^2}{4^2} = -2.5 \text{ kN.m} \qquad (1)$$

Note that the frame is restrained against sidesway. The spans must be considered for writing slope-deflection equations viz, A , B and AC . The beam end moments are related to unknown rotations θ_A and θ_B by following slope-deflection equations. (Force deflection equations). Support C is fixed and hence $\theta_C = 0$.

$$M_{AB} = M_{AB}^{F} + \frac{2E(2I)}{L_{AB}}(2\theta_A + \theta_B)$$

$$M_{AB} = 15 - +1.333EI\theta_A + 0.667EI\theta_B$$

$$M_{BA} = -15 + 0.667EI\theta_A + 1.333EI\theta_B$$

$$M_{BC} = 2.5 + EI\theta_B + 0.5EI\theta_C$$

$$M_{CB} = -2.5 + 0.5EI\theta_B \qquad (2)$$

Consider the joint equilibrium of support *A* (See Fig 16.4 (c))

$$\sum M_A = 0$$

$$M_{AB} = 0 = 15 + 1.333EI\theta_A + 0.667EI\theta_B \qquad (3)$$

$$1.333EI\theta_A + +0.667EI\theta_B = -15$$

Or, $2\theta_A + \theta_B = \dfrac{-22.489}{EI}$

Equilibrium of joint *B* (Fig 16.4(d))

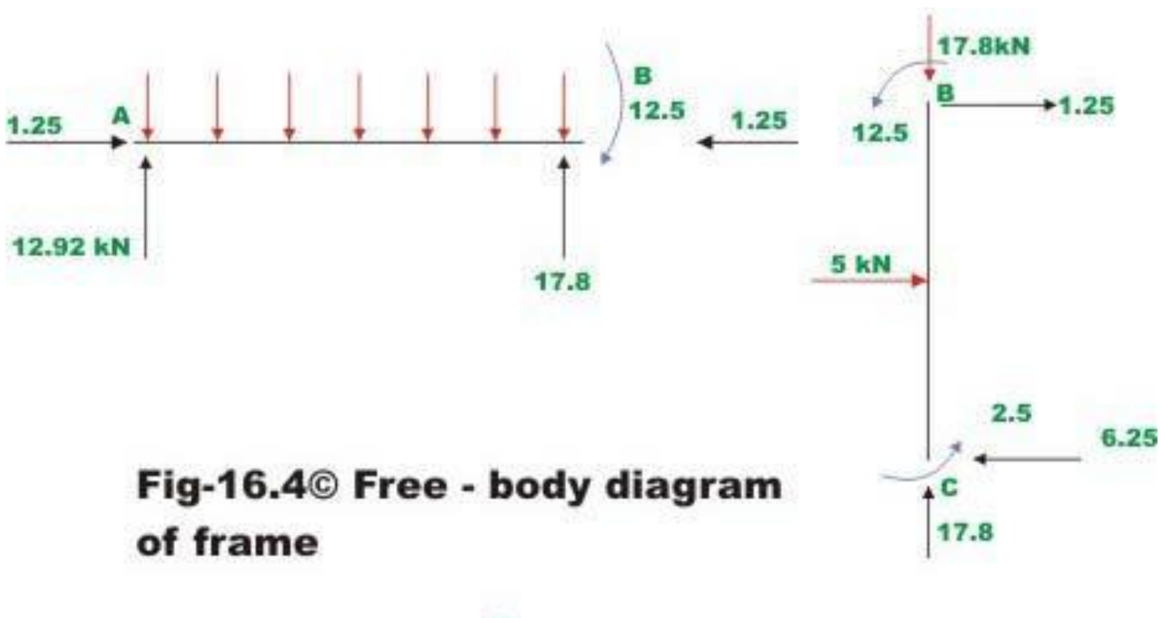


Fig-16.4© Free - body diagram of frame

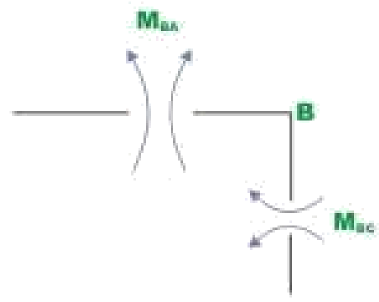


Fig-16.4(d) Free - body diagram of joint B

$$\sum M_B = 0 \quad \Rightarrow \quad M_{BC} + M_{BA} = 0 \qquad (4)$$

Substituting the value of M_{BC} and M_{BA} in the above equation,

$$2.333EI\theta_B + 0.667EI\theta_A = 12.5 \qquad (5)$$

Or, $$3.498\theta_B + \theta_A = \frac{18.741}{EI}$$

Solving equation (3) and (4)

$$\theta_B = \frac{10.002}{EI}\ (counterclockwise)$$

$$\theta_B = \frac{-16.245}{EI}\ (clockwise) \qquad (6)$$

Substituting the value of θ_A and θ_B in equation (2) beam end moments are evaluated.

$$M_{AB} = 15 + 1.333EI\left(\frac{-16.245}{EI}\right) + 0.667EI\left(\frac{10.002}{EI}\right) = 0$$

$$M_{BA} = -15 + 0.667EI\left(\frac{-16.245}{EI}\right) + .1.33EI\left(\frac{10.002}{EI}\right) = -1$$

$$M_{BC} = 2.5 + EI\left(\frac{10.002}{EI}\right) = 12.5 \text{ kN.m}$$

$$M_{CB} = -2.5 + 0.5EI\left(\frac{10.002}{EI}\right) = 2.5 \text{ kN.m} \qquad (7)$$

Using these results, reactions are evaluated from equilibrium equations as shown in Fig 16.4 (e)

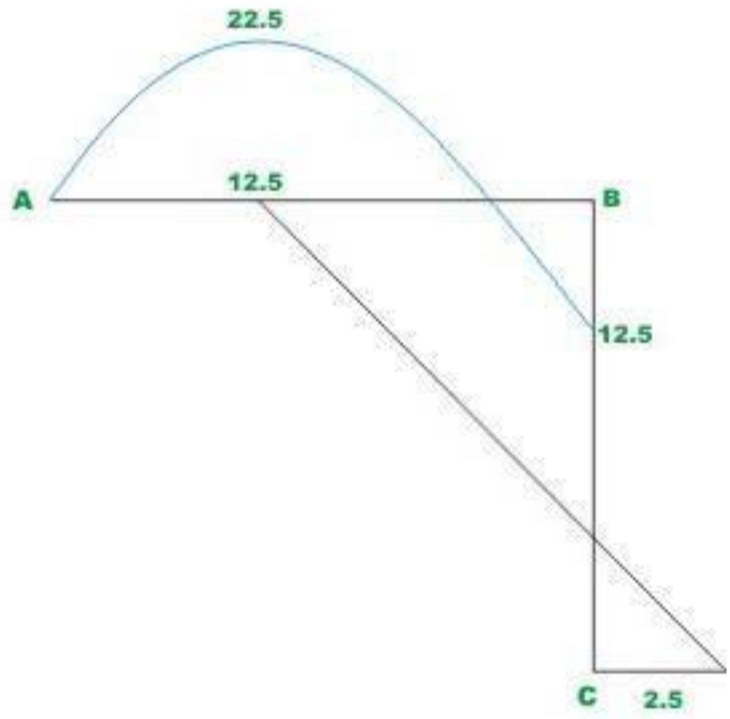


Fig-16.4(e) B.M.D

The shear force and bending moment diagrams are shown in Fig 16.4(g) and 16.4 h respectively. The qualitative elastic curve is shown in Fig 16.4 (h).

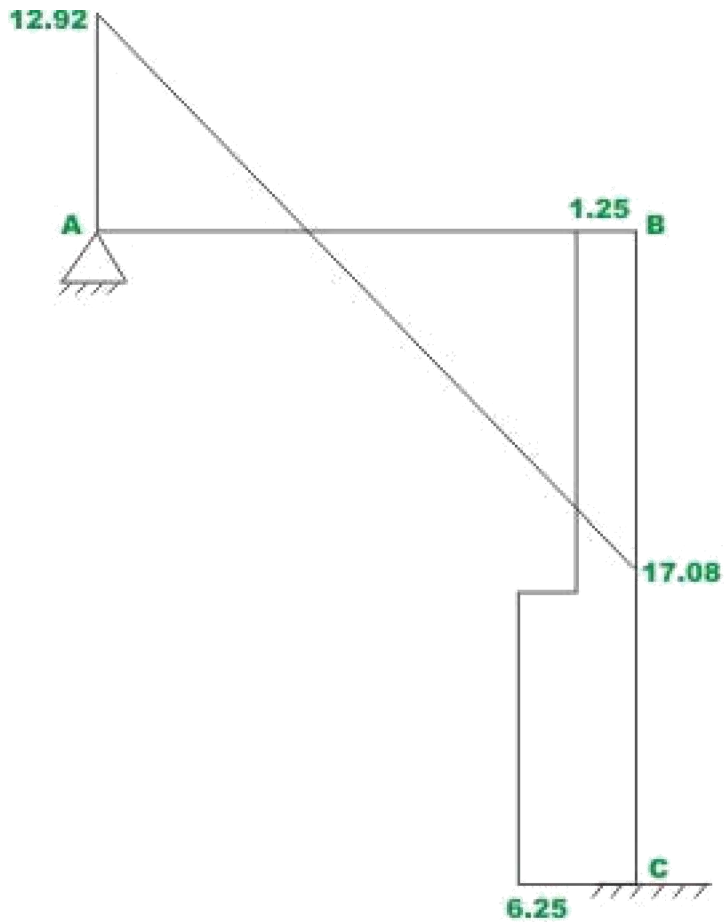


Fig-16.4(f) S.F.D

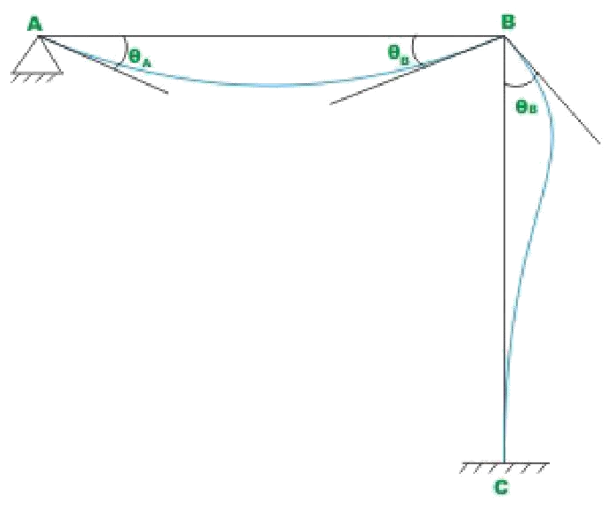


Fig.16.4 (g)Elastic Curve

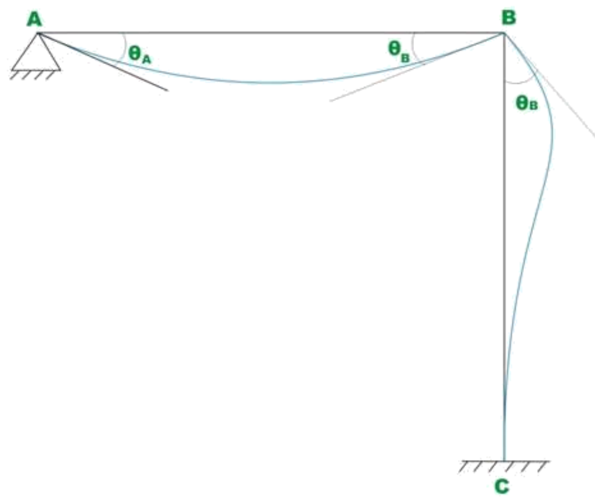


Fig- 16.4(h) Elastic Curve

Example

Compute reactions and beam end moments for the rigid frame shown in Fig 16.5(a). Draw bending moment diagram and sketch the elastic curve for the frame.

Solution

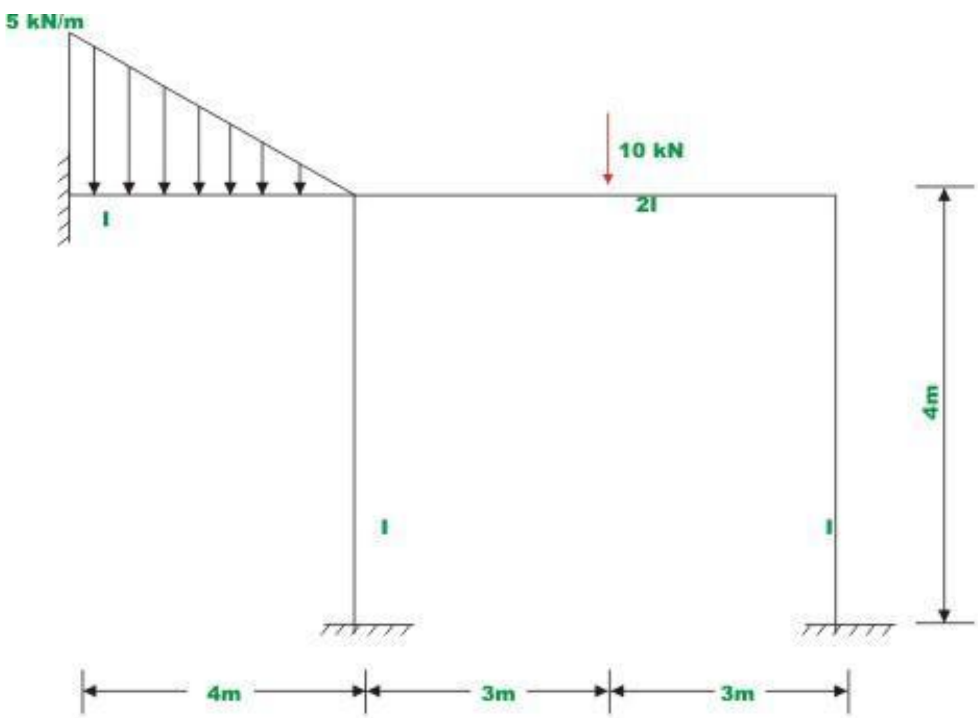


Fig-16.5(a) Example 16.3

The given frame is kinematically indeterminate to third degree so three rotations are to be calculated, θ_B, θ_C and θ_D. First calculate the fixed end moments (see Fig 16.5 b).

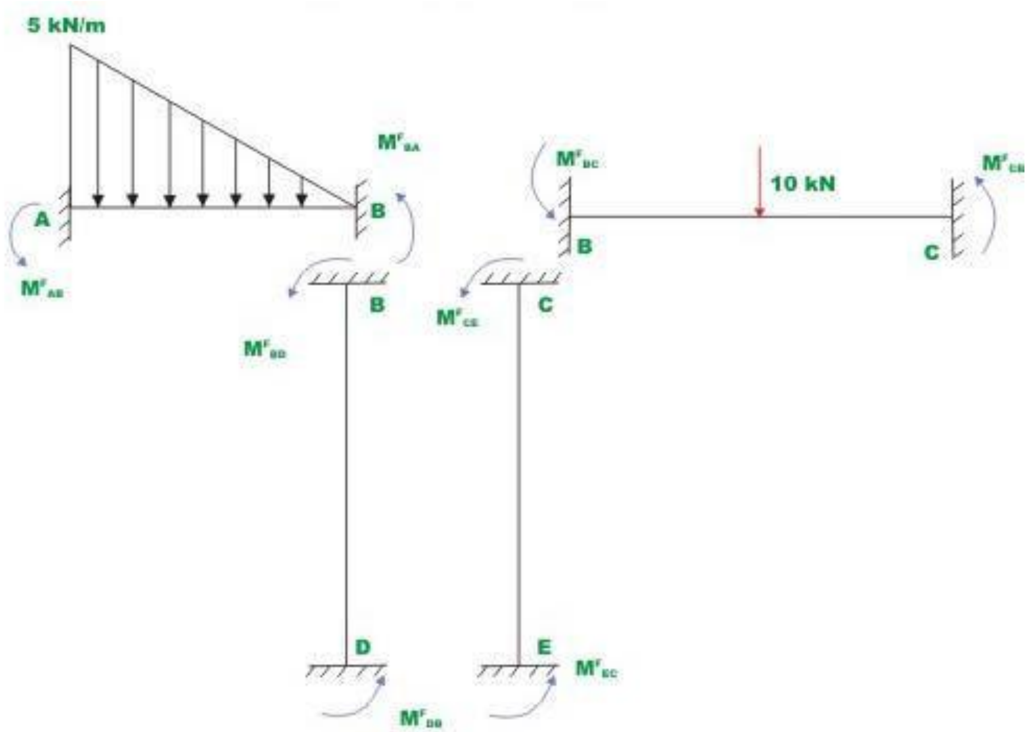


Fig.16.5b Kinematically restrained structure

$$M_{AB}^F = \frac{5 \times 4^2}{20} = 4 \text{ kN.m}$$

$$M_{BA}^F = \frac{-5 \times 4^2}{30} = -2.667 \text{ kN.m}$$

$$M_{BC}^F = \frac{10 \times 3 \times 3^2}{6^2} = 7.5 \text{ kN.m}$$

$$M_{CB}^F = \frac{-10 \times 3 \times 3^2}{6^2} = -7.5 \text{ kN.m}$$

$$M_{BD}^F = M_{DB}^F = M_{CE}^F = M_{EC}^F = 0 \qquad (1)$$

The frame is restrained against sidesway. Four spans must be considered for rotating slope – deflection equation: AB, BD, BC and CE. The beam end moments are related to unknown rotation at B, C, and D. Since the supports A and E are fixed. $\theta_A = \theta_E = 0$.

$$M_{AB} = 4 + \frac{2EI}{4}[2\theta_A + \theta_B]$$

$$M_{AB} = 4 + EI\,\theta_A + 0.5\,EI\,\theta_B = 4 + 0.5EI\theta_B$$

$$M_{BA} = -2.667EI\,\theta_A + EI\theta_B = -2.667 + EI\theta_B$$

$$M_{BD} = EI\,\theta_B + 0.5EI\theta_D$$

$$M_{DB} = 0.5EI\,\theta_B + EI\theta_D$$

$$M_{BC} = 7.5 + \frac{2E(2I)}{6}[2\theta_B + \theta_C] = 7.5 + 1.333\,EI\,\theta_B + 0.667EI\theta_C$$

$$M_{CB} = -7.5 + .667\,EI\,\theta_B + 1.333EI\theta_C$$

$$M_{CE} = EI\,\theta_C + 0.5EI\,\theta_E = EI\theta_C$$

$$M_{EC} = 0.5\,EI\,\theta_C + 0.5\,EI\,\theta_E = 0.5EI\theta_C \qquad (2)$$

Consider the equilibrium of joints B, D, C (vide Fig. 16.5(c))

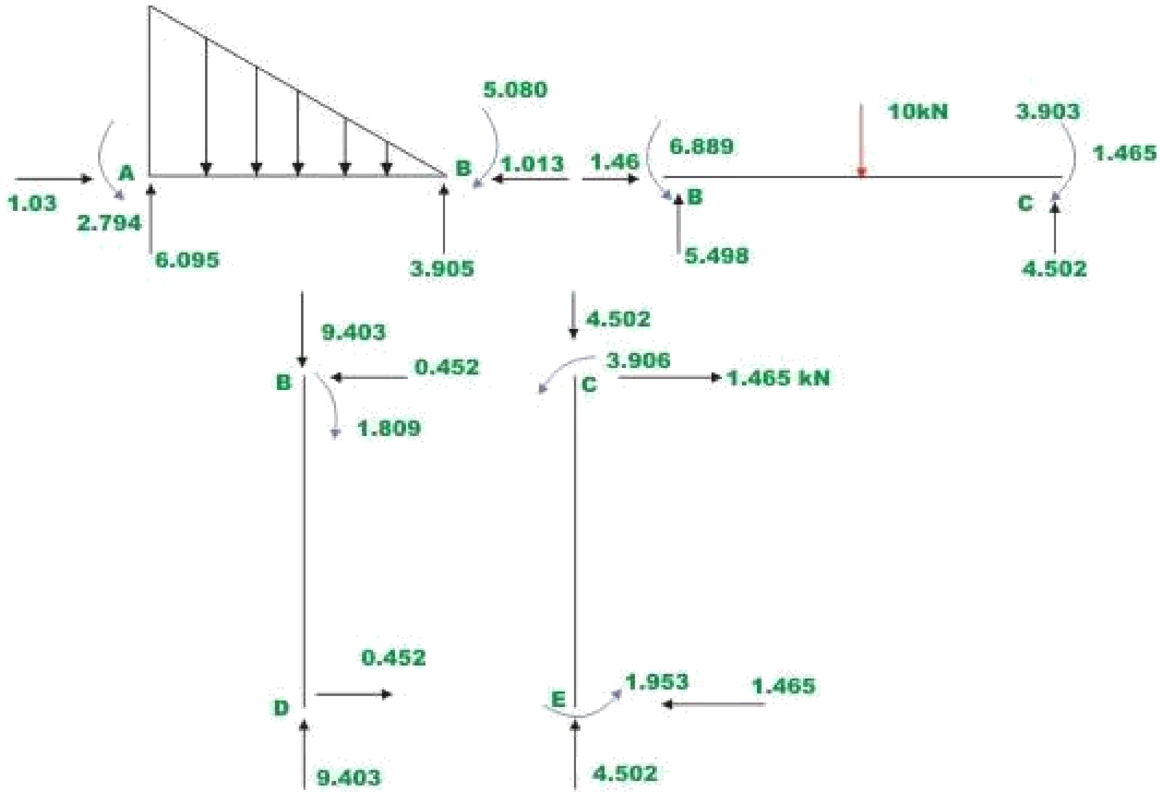


Fig-16.5 (c) Free - body diagram

$$\sum M_B = 0 \Rightarrow M_{BA} + M_{BC} + M_{BD} = 0 \quad (3)$$

$$\sum M_D = 0 \Rightarrow M_{DB} = 0 \quad (4)$$

$$\sum M_C = 0 \Rightarrow M_{CB} + M_{CE} = 0 \quad (5)$$

Substituting the values of M_{BA} , M_{BC} , M_{BD} , M_{DB} , M_{CB} and M_{CE} in the equations (3), (4), and (5)

$$3.333\, EI\, \theta_B + 0.667\, EI\, \theta_C + 0.5\, EI\theta_D = -4.833$$

$$0.5EI\, \theta_B + EI\theta_D = 0$$

$$2.333\, EI\, \theta_C + 0.667\, EI\theta_B = 7.5 \quad (6)$$

Solving the above set of simultaneous equations, θ_B ,θ_C and θ_D are evaluated.

$$EI\theta_B = -2.4125$$

$$EI\theta_C = 3.9057$$

$$EI\theta_D = 1.2063 \quad (7)$$

Substituting the values of θ_B ,θ_C and θ_D in (2), beam end moments are computed.

$$M_{AB} = 2.794 \text{ kN.m}$$

$$M_{BA} = -5.080 \text{ kN.m}$$

$$M_{BD} = -1.8094 \text{ kN.m}$$

$$M_{DB} = 0$$

$$M_{BC} = 6.859 \text{ kN.m}$$

$$M_{CB} = -3.9028 \text{ kN.m}$$

$$M_{CE} = 3.9057 \text{ kN.m}$$

$$M_{EC} = 1.953 \text{ kN.m} \quad (8)$$

The reactions are computed in Fig 16.5(d), using equilibrium equations known beam-end moments and given loading.

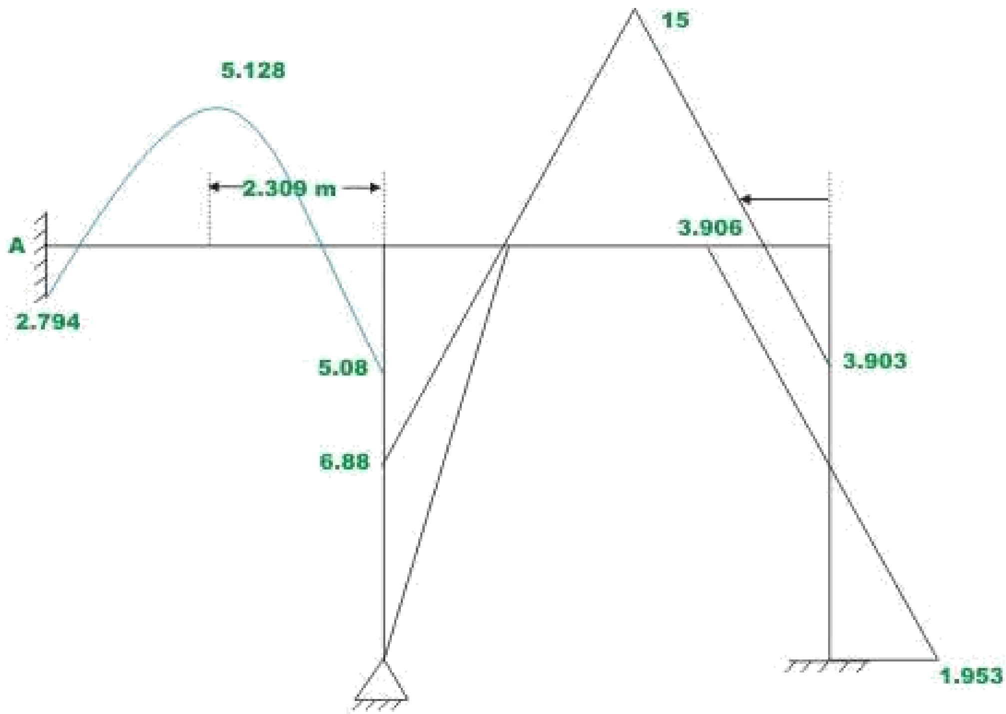


Fig-16.5(d) Bending moment diagram

$R_{Ay} = 6.095$ kN $(\uparrow)$

$R_{Dy} = 9.403$ kN $(\uparrow)$

$R_{Ey} = 4.502$ kN $(\uparrow)$

$R_{Ax} = 1.013$ kN $(\rightarrow)$

$R_{Dx} = 0.542$ kN $(\rightarrow)$

$R_{Ex} = -1.465$ kN $(\leftarrow)$ (9)

The bending moment diagram is shown in Fig 16.5.(e) and the elastic curve is shown in Fig 16.5(f).

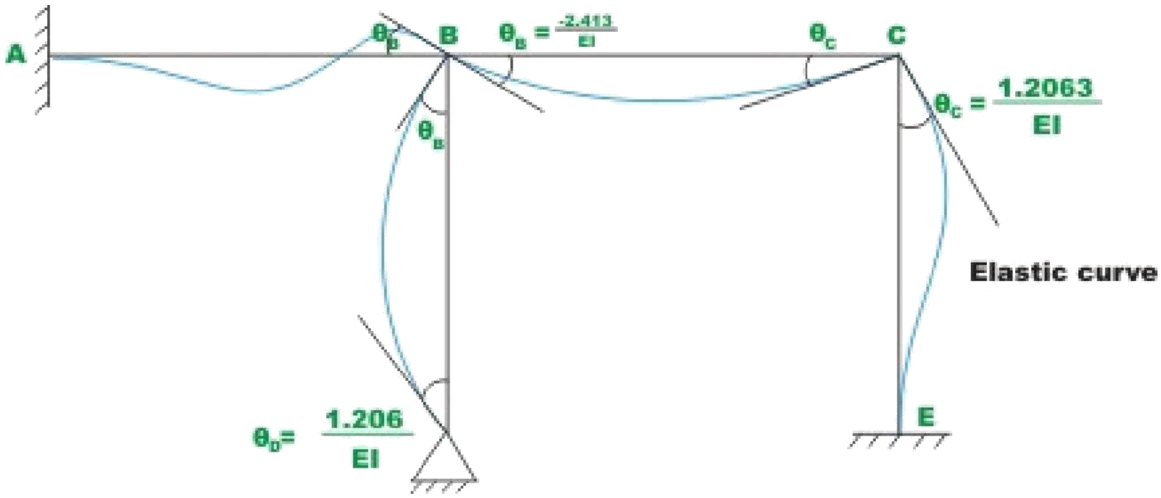


Fig-16.5(e)

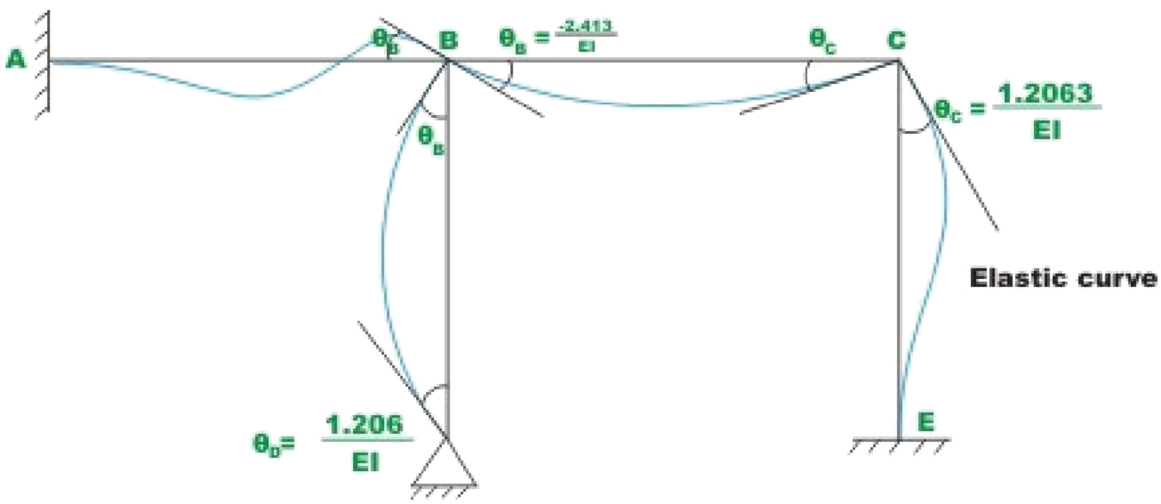


Fig-16.5(f)

Objectives

After reading this chapter the student will be able to
1. Derive slope-deflection equations for the frames undergoing sidesway.
2. Analyse plane frames undergoing sidesway.
3, Draw shear force and bending moment diagrams.
4. Sketch deflected shape of the plane frame not restrained against sidesway.

Introduction

In this lesson, slope-deflection equations are applied to analyse statically indeterminate frames undergoing sidesway. As stated earlier, the axial deformation of beams and columns are small and are neglected in the analysis. In the previous lesson, it was observed that sidesway in a frame will not occur if

1. They are restrained against sidesway.
2. If the frame geometry and the loading are symmetrical.

In general loading will never be symmetrical. Hence one could not avoid sidesway in frames.

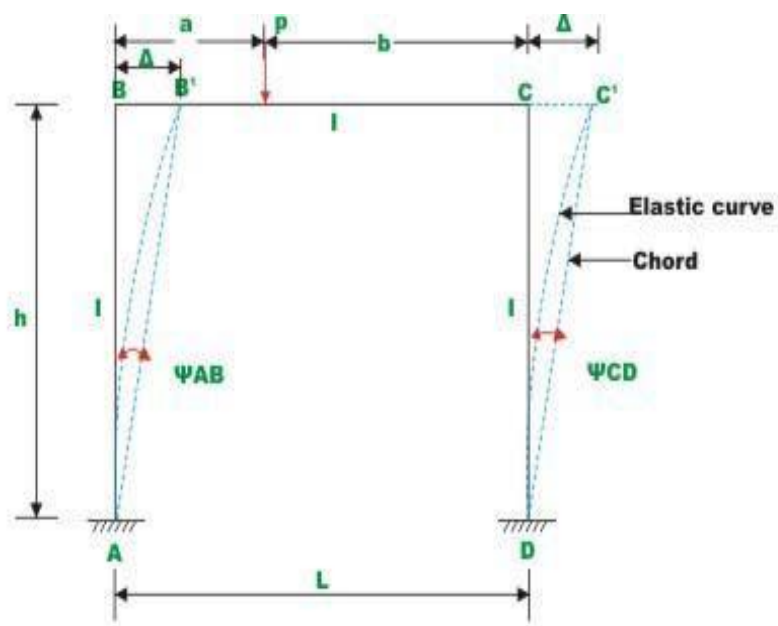


Fig.17.1 Plane frame undergoing sway

For example, consider the frame of Fig. 17.1. In this case the frame is symmetrical but not the loading. Due to unsymmetrical loading the beam end moments M_{BC} and M_{CB} are not equal. If b is greater than a, then $M_{BC} > M_{CB}$. In such a case joint B and C are displaced toward right as shown in the figure by an unknown amount . Hence we have three unknown displacements in this frame: rotations θ_B ,θ_C and the linear displacement . The unknown joint rotations θ_B and θ_C are related to joint moments by the moment equilibrium equations. Similarly, when unknown linear displacement occurs, one needs to consider force-equilibrium equations. While applying slope-deflection equation to columns

unknowns. It is observed that in the column AB , the end B undergoes a linear displacement with respect to end A . Hence the slope-deflection equation for column AB is similar to the one for beam undergoing support settlement. However, in this case is unknown. For each of the members we can write the following slope-deflection equations.

$$M_{AB} = M_{AB}^F + \frac{2EI}{h}\left[2\theta_A + \theta_B - 3\psi_{AB}\right] \qquad \text{where } \psi_{AB} = -\frac{}{h}$$

ψ_{AB} is assumed to be negative as the chord to the elastic curve rotates in the clockwise directions.

$$M_{BA} = M_{BA}^F + \frac{2EI}{h}\left[2\theta_B + \theta_A - 3\psi_{AB}\right]$$

$$M_{BC} = M_{BC}^F + \frac{2EI}{h}\left[2\theta_B + \theta_C\right]$$

$$M_{CB} = M_{CB}^F + \frac{2EI}{h}\left[2\theta_C + \theta_B\right]$$

$$M_{CD} = M_{CD}^F + \frac{2EI}{h}\left[2\theta_C + \theta_D - 3\psi_{CD}\right] \qquad \psi_{CD} = -\frac{}{h}$$

$$M_{DC} = M_{DC}^F + \frac{2EI}{h}\left[2\theta_D + \theta_C - 3\psi_{CD}\right] \tag{17.1}$$

As there are three unknowns (θ_B ,θ_C and), three equations are required to evaluate them. Two equations are obtained by considering the moment equilibrium of joint B and C respectively.

$$\sum M_B = 0 \qquad M_{BA} + M_{BC} = 0 \tag{17.2a}$$

$$\sum M_C = 0 \qquad M_{CB} + M_{CD} = 0 \tag{17.2b}$$

Now consider free body diagram of the frame as shown in Fig. 17.2. The horizontal shear force acting at A and B of the column AB is given by

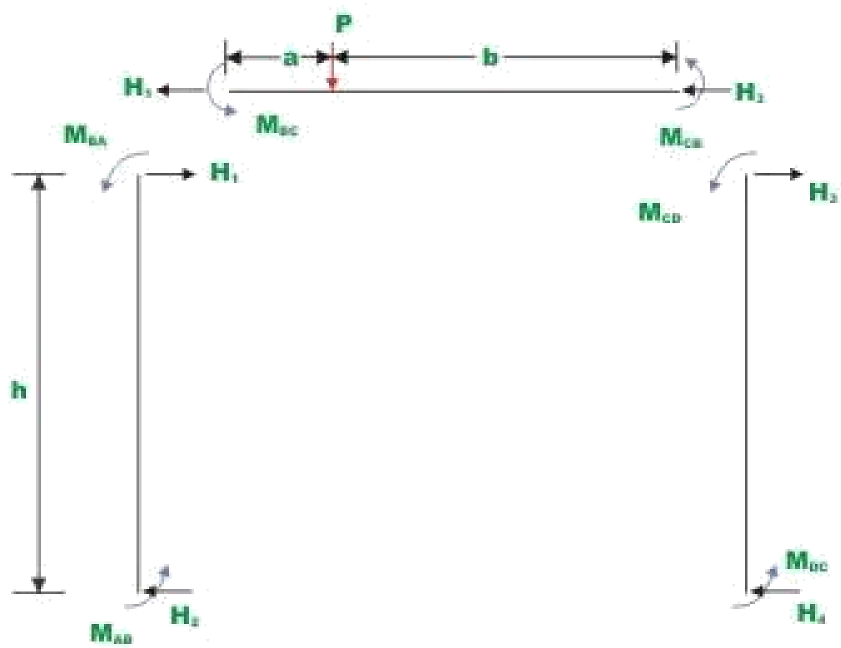


Fig.17.2 Free - body diagrams of columns and beams

$$H_1 = \frac{M_{BA} + M_{AB}}{h} \quad (17.3a)$$

Similarly for member CD , the shear force H_3 is given by

$$H_3 = \frac{M_{CD} + M_{DC}}{h} \quad (17.3b)$$

Now, the required third equation is obtained by considering the equilibrium of member BC ,

$$\sum F_X = 0 \qquad \Rightarrow H_1 + H_3 = 0$$

$$\frac{M_{BA} + M_{AB}}{h} + \frac{M_{CD} + M_{DC}}{h} = 0 \quad (17.4)$$

Substituting the values of beam end moments from equation (17.1) in equations (17.2a), (17.2b) and (17.4), we get three simultaneous equations in three unknowns θ_B ,θ_C and , solving which joint rotations and translations are evaluated.

Knowing joint rotations and translations, beam end moments are calculated from slope-deflection equations. The complete procedure is explained with a few numerical examples.

Example

Analyse the rigid frame as shown in Fig. 17.3a. Assume EI to be constant for all members. Draw bending moment diagram and sketch qualitative elastic curve.

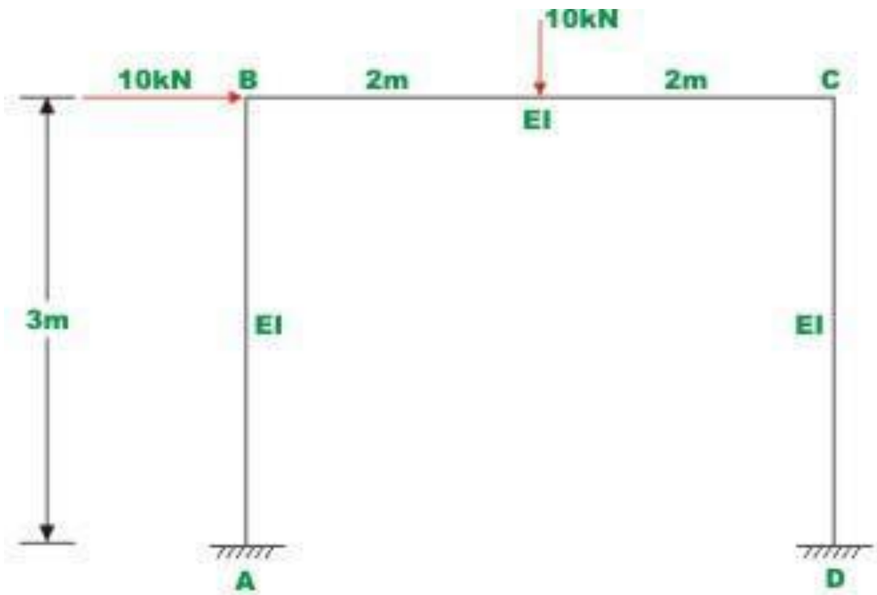


Fig.17.3 (a) Example 17.1

Solution

In the given problem, joints B and C rotate and also translate by an amount . Hence, in this problem we have three unknown displacements (two rotations and one translation) to be evaluated. Considering the kinematically determinate structure, fixed end moments are evaluated. Thus,

$$M_{AB}^{F} = 0\,;\; M_{BA}^{F} = 0\,;\; M_{BC}^{F} = +10\ kN.m\,;\; M_{CB}^{F} = -10 kN.m\,;\; M_{CD}^{F} = 0\,;\; M_{DC}^{F} = 0. \qquad (1)$$

The ends A and D are fixed. Hence, $\theta_A = \theta_D = 0.$ Joints B and C translate by the same amount . Hence, chord to the elastic curve AB' and DC' rotates by an amount (see Fig. 17.3b)

$$\psi_{AB} = \psi_{CD} = -\frac{}{3} \qquad (2)$$

Chords of the elastic curve AB' and DC' rotate in the clockwise direction; hence ψ_{AB} and ψ_{CD} are taken as negative.

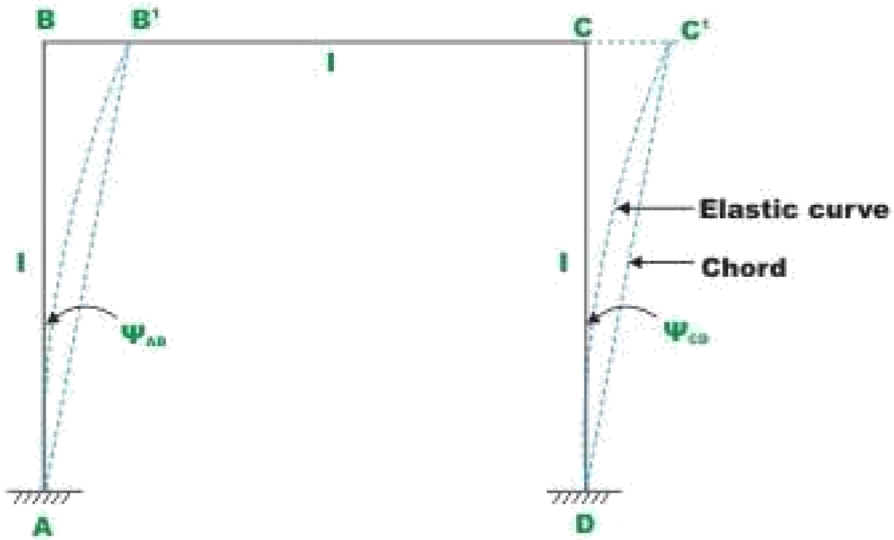


Fig.17.3b Column ratation

Now, writing the slope-deflection equations for the six beam end moments,

$$M_{AB} = M_{AB}^F + \frac{2EI}{3}\left[2\theta_A + \theta_B - 3\psi_{AB}\right]$$

$$M_{AB}^F = 0\,; \theta_A = 0\,; \psi_{AB} = -\frac{}{3}\,.$$

$$M_{AB} = \frac{2}{3} EI\theta_B + \frac{2}{3} EI$$

$$M_{BA} = \frac{4}{3} EI\theta_B + \frac{2}{3} EI$$

$$M_{BC} = 10 + EI\theta_B + \frac{1}{2} EI\theta_C$$

$$M_{CB} = -10 + \frac{1}{2} EI\theta_B + EI\theta_C$$

$$M_{CD} = \frac{4}{3} EI\theta_C + \frac{2}{3} EI$$

$$M_{DC} = \frac{2}{3} EI\theta_C + \frac{2}{3} EI \quad (3)$$

Now, consider the joint equilibrium of B and C (vide Fig. 17.3c).

$$\sum M_B = 0 \quad \Rightarrow \quad M_{BA} + M_{BC} = 0 \quad (4)$$

$$\sum M_C = 0 \quad \Rightarrow \quad M_{CB} + M_{CD} = 0 \quad (5)$$

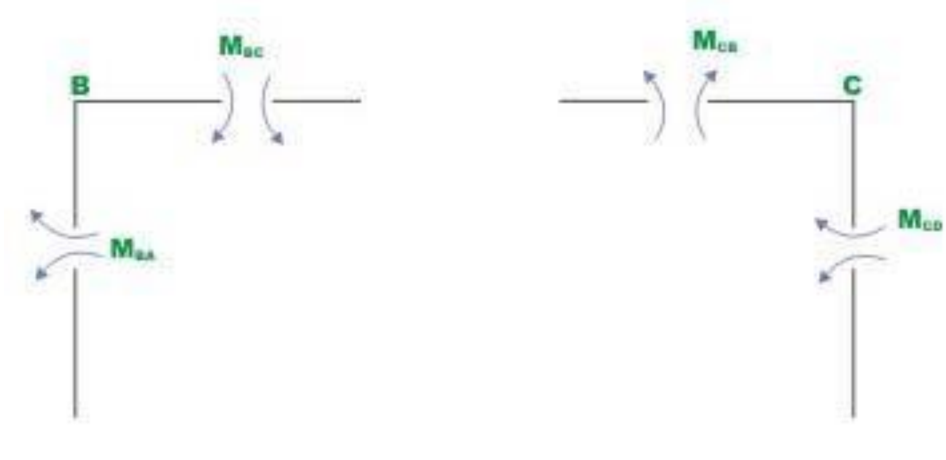


Fig.17.3c Free - body diagram of joints B and C

The required third equation is written considering the horizontal equilibrium of the entire frame *i.e.* $\sum F_X = 0$ (vide Fig. 17.3d).

$$-H_1 + 10 - H_2 = 0$$

$$\Rightarrow H_1 + H_2 = 10 . \quad (6)$$

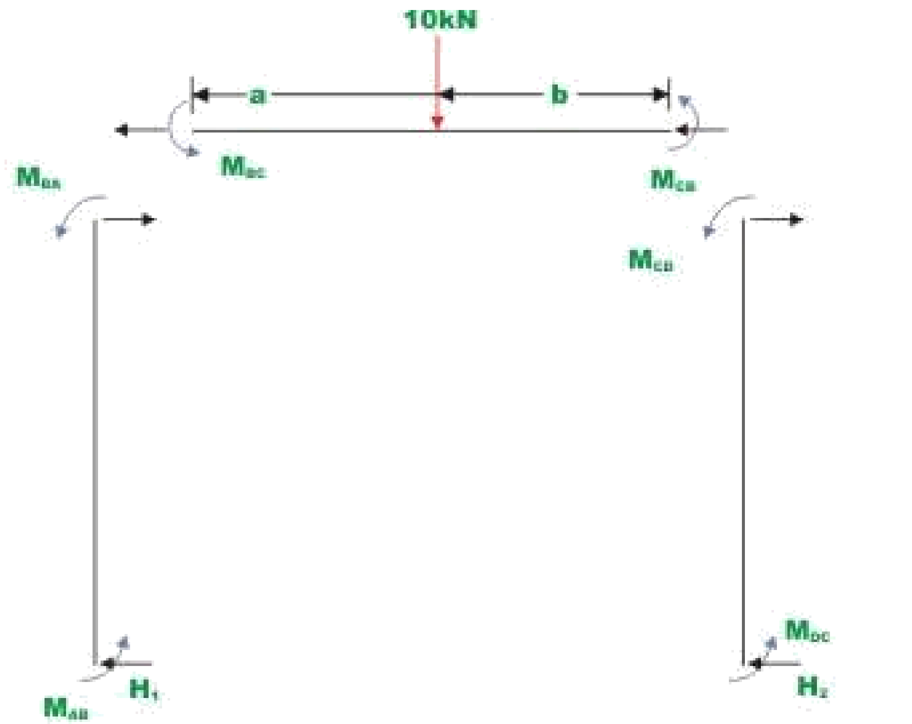


Fig.17.3d Free - body diagram of frame

Considering the equilibrium of the column AB and CD , yields

$$H_1 = \frac{M_{BA} + M_{AB}}{3}$$

and

$$H_2 = \frac{M_{CD} + M_{DC}}{3} \qquad (7)$$

The equation (6) may be written as,

$$M_{BA} + M_{AB} + M_{CD} + M_{DC} = 30 \qquad (8)$$

Substituting the beam end moments from equation (3) in equations (4), (5) and (6)

$$2.333EI\theta_B + 0.5EI\theta_C + 0.667EI = -10 \qquad (9)$$

$$2.333EI\theta_C + 0.5EI\theta_B + 0.667EI = 10 \qquad (10)$$

$$2EI\theta_B + 2EI\theta_C + \frac{8}{3}EI = 30 \quad (11)$$

Equations (9), (10) and (11) indicate symmetry and this fact may be noted. This may be used as the check in deriving these equations.

Solving equations (9), (10) and (11),

$EI\theta_B = -9.572$; $EI\theta_C = 1.355$ and $EI = 17.417$.

Substituting the values of $EI\theta_B$, $EI\theta_C$ and EI in the slope-deflection equation (3), one could calculate beam end moments. Thus,

$$M_{AB} = 5.23 \text{ kN.m (counterclockwise)}$$

$$M_{BA} = -1.14 \text{ kN.m(clockwise)}$$

$$M_{BC} = 1.130 \text{ kN.m}$$

$$M_{CB} = -13.415 \text{ kN.m}$$

$$M_{CD} = 13.406 \text{ kN.m}$$

$$M_{DC} = 12.500 \text{ kN.m} .$$

The bending moment diagram for the frame is shown in Fig. 17.3 e. And the elastic curve is shown in Fig 17.3 f. the bending moment diagram is drawn on the compression side. Also note that the vertical hatching is used to represent bending moment diagram for the horizontal members (beams).

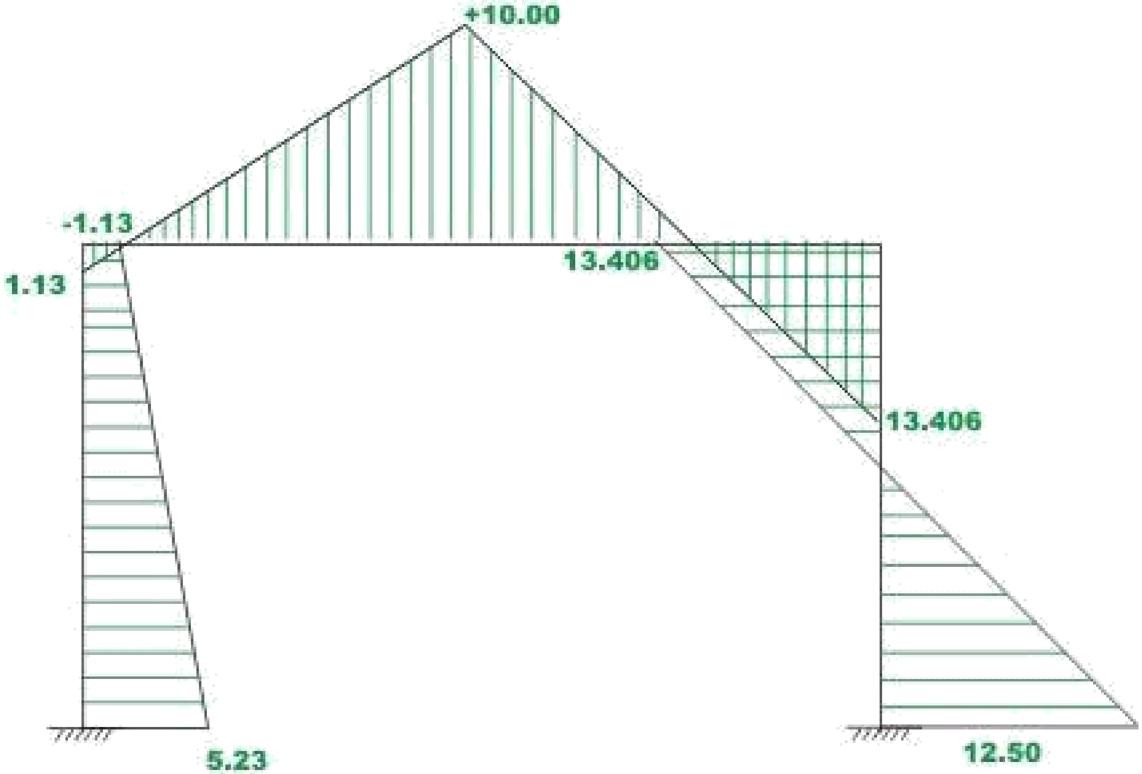


Fig.17.3e Bending moment diagram

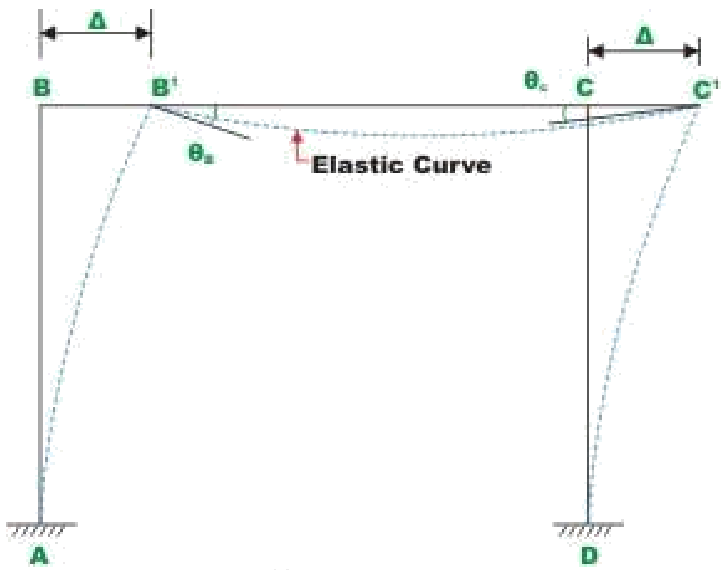


Fig.17.3f Elastic curve

Example 2

Analyse the rigid frame as shown in Fig. 17.4a and draw the bending moment diagram. The moment of inertia for all the members is shown in the figure. Neglect axial deformations.

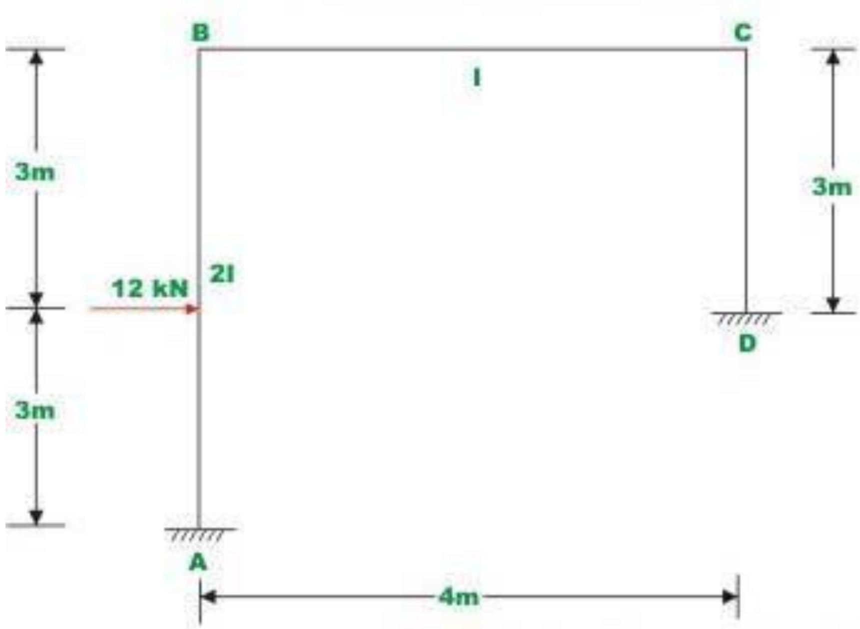


Fig.17.4a (Example 17.2)

Solution:

In this problem rotations and translations at joints B and C need to be evaluated. Hence, in this problem we have three unknown displacements: two rotations and one translation. Fixed end moments are

$$M_{AB}^{F} = \frac{12 \times 3 \times 9}{36} = 9\ kN.m\ ;\ M_{BA}^{F} = -9\ kN.m\ ; \qquad (1)$$
$$M_{BC}^{F} = 0\ ;\ M_{CB}^{F} = 0\ ;\ M_{CD}^{F} = 0\ ;\ M_{DC}^{F} = 0.$$

The joints B and C translate by the same amount . Hence, the chord to the elastic curve rotates in the clockwise direction as shown in Fig. 17.3b.

$$\psi_{AB} = -\frac{\ }{6}$$

and

$$\psi_{CD} = -\frac{\ }{3} \qquad (2)$$

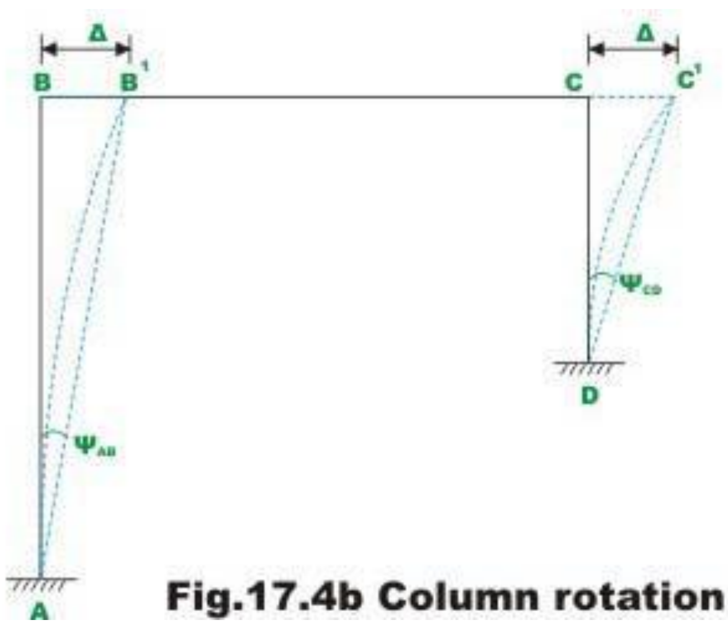


Fig.17.4b Column rotation due to sway

Now, writing the slope-deflection equations for six beam end moments,

$$M_{AB} = 9 + \frac{2(2EI)}{6}\theta_B + 2$$

$$M_{AB} = 9 + 0.667EI\theta_B + 0.333EI$$

$$M_{BA} = -9 + 1.333EI\theta_B + 0.333EI$$

$$M_{BC} = EI\theta_B + 0.5EI\theta_C$$

$$M_{CB} = 0.5EI\theta_B + EI\theta_C$$

$$M_{CD} = 1.333EI\theta_C + 0.667EI$$

$$M_{DC} = 0.667EI\theta_C + 0.667EI \qquad (3)$$

Now, consider the joint equilibrium of B and C.

$$\sum M_B = 0 \quad \Rightarrow \quad M_{BA} + M_{BC} = 0 \qquad (4)$$

$$\sum M_C = 0 \quad \Rightarrow \quad M_{CB} + M_{CD} = 0 \qquad (5)$$

The required third equation is written considering the horizontal equilibrium of the entire frame. Considering the free body diagram of the member BC (vide Fig. 17.4c),

$H_1 + H_2 = 0$.

(6)

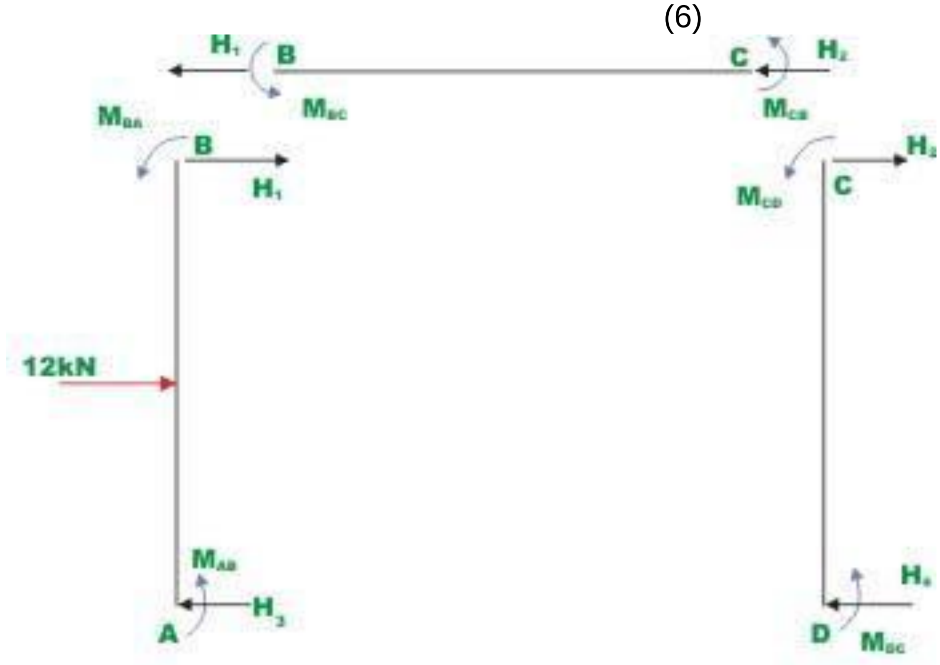


Fig.17.4c Free - body diagram

The forces H_1 and H_2 are calculated from the free body diagram of column AB and CD. Thus,

$$H_1 = -6 + \frac{M_{BA} + M_{AB}}{6}$$

and

$$H_2 = \frac{M_{CD} + M_{DC}}{3} \qquad (7)$$

Substituting the values of H_1 and H_2 into equation (6) yields,

$$M_{BA} + M_{AB} + 2M_{CD} + 2M_{DC} = 36 \qquad (8)$$

Substituting the beam end moments from equation (3) in equations (4), (5) and (8), yields

$$2.333EI\theta_B + 0.5EI\theta_C + 0.333EI = 9$$

$$2.333EI\theta_C + 0.5EI\theta_B + 0.667EI = 0$$

$$2EI\theta_B + 4EI\theta_C + 3.333EI = 36 \qquad (9)$$

Solving equations (9), (10) and (11),

$$EI\theta_B = 2.76\,; \quad EI\theta_C = -4.88 \quad \text{and} \quad EI = 15.00.$$

Substituting the values of $EI\theta_B$, $EI\theta_C$ and EI in the slope-deflection equation (3), one could calculate beam end moments. Thus,

$$M_{AB} = 15.835 \text{ kN.m} \quad \text{(counterclockwise)}$$
$$M_{BA} = -0.325 \text{ kN.m(clockwise)}$$
$$M_{BC} = 0.32 \text{ kN.m}$$
$$M_{CB} = -3.50 \text{ kN.m}$$
$$M_{CD} = 3.50 \text{ kN.m}$$
$$M_{DC} = 6.75 \text{ kN.m} .$$

The bending moment diagram for the frame is shown in Fig. 17.4 d.

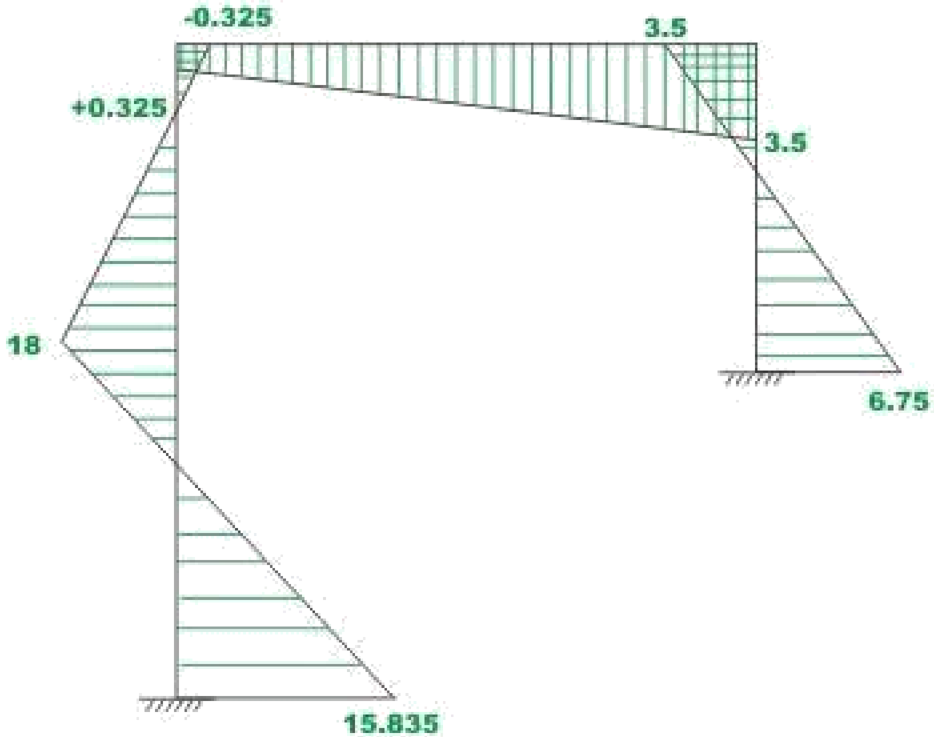


Fig.17.4d Bending moment diagram

Example 3

Analyse the rigid frame shown in Fig. 17.5 a. Moment of inertia of all the members are shown in the figure. Draw bending moment diagram.

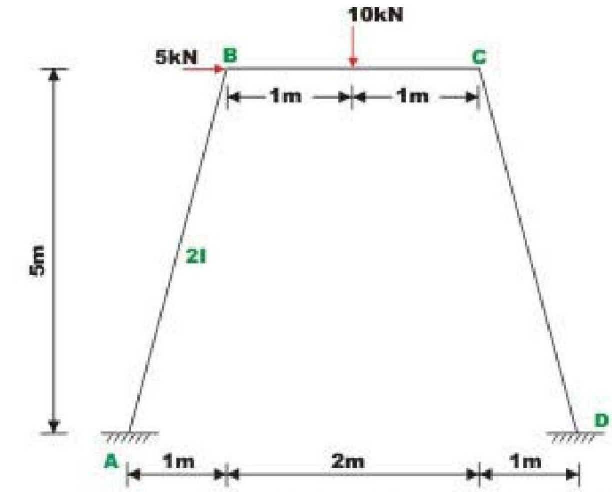


Fig.17.5a Example 17.3

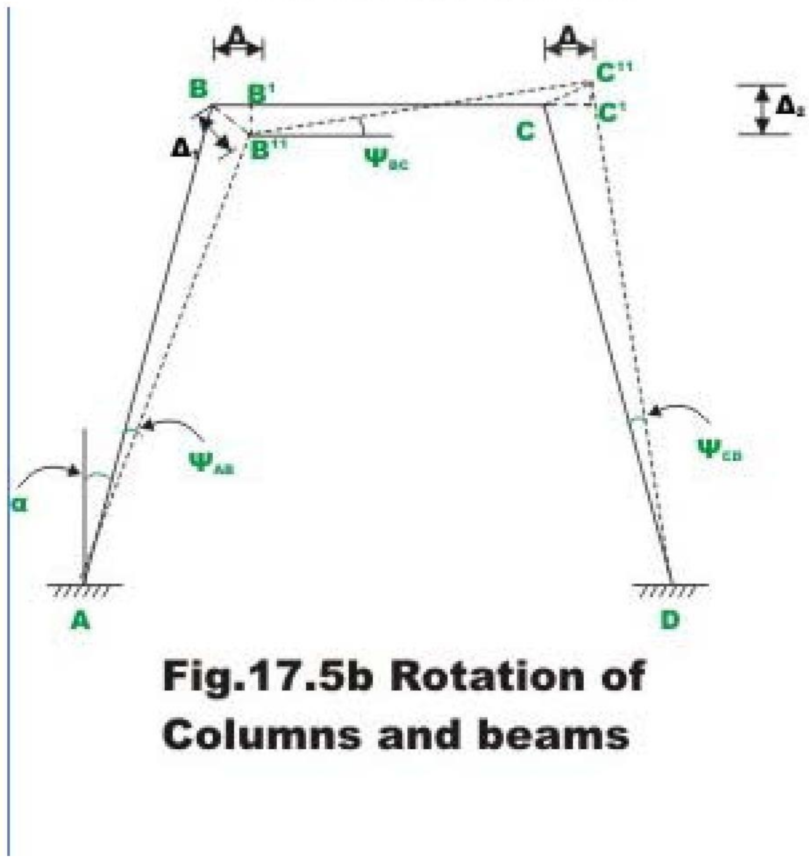


Fig.17.5b Rotation of Columns and beams

Under the action of external forces, the frame gets deformed as shown in Fig. 17.5b. In this figure, chord to the elastic curve are shown by dotted line. BB' is perpendicular to AB and CC'' is perpendicular to DC . The chords to the elastic curve AB'' rotates by an angle ψ_{AB} , $B''C''$ rotates by ψ_{BC} and DC rotates by ψ_{CD} as shown in figure. Due to symmetry, $\psi_{CD} = \psi_{AB}$. From the geometry of the figure,

$$\psi_{AB} = \frac{BB''}{L_{AB}} = -\frac{1}{L_{AB}}$$

But

$$_1 = \frac{\quad}{\cos\alpha}$$

Thus,

$$\psi_{AB} = -\frac{\quad}{L_{AB}\cos\alpha} = -\frac{\quad}{5}$$

$$\psi_{CD} = -\frac{\quad}{5}$$

$$\psi_{BC} = \frac{2}{2} = \frac{2\tan\alpha}{2} = \tan\alpha = \frac{\quad}{5} \qquad (1)$$

We have three independent unknowns for this problem θ_B , θ_C and . The ends A and D are fixed. Hence, $\theta_A = \theta_D = 0$. Fixed end moments are,

$M_{AB}^F = 0$; $M_{BA}^F = 0$; $M_{BC}^F = +2.50\ kN.m$; $M_{CB}^F = -2.50 kN.m$; $M_{CD}^F = 0$; $M_{DC}^F = 0$.

Now, writing the slope-deflection equations for the six beam end moments,

$$M_{AB} = \frac{2E(2I)}{5.1}\left[\theta_A - 3\psi_{AB}\right]$$

$$M_{AB} = 0.784 EI\theta_B + 0.471 EI$$

$$M_{BA} = 1.568 EI\theta_B + 0.471 EI$$

$$M_{BC} = 2.5 + 2EI\theta_B + EI\theta_C - 0.6EI$$

$$M_{BC} = -2.5 + EI\theta_B + 2EI\theta_C - 0.6EI$$

$$M_{CD} = 1.568 EI\theta_C + 0.471 EI$$

$$M_{DC} = 0.784 EI\theta_C + 0.471 EI \qquad (2)$$

Now, considering the joint equilibrium of B and C , yields

$$\sum M_B = 0 \quad \Rightarrow M_{BA} + M_{BC} = 0$$

$$3.568EI\theta_B + EI\theta_C - 0.129EI = -2.5 \tag{3}$$

$$\sum M_C = 0 \quad \Rightarrow M_{CB} + M_{CD} = 0$$

$$3.568EI\theta_C + EI\theta_B - 0.129EI = 2.5 \tag{4}$$

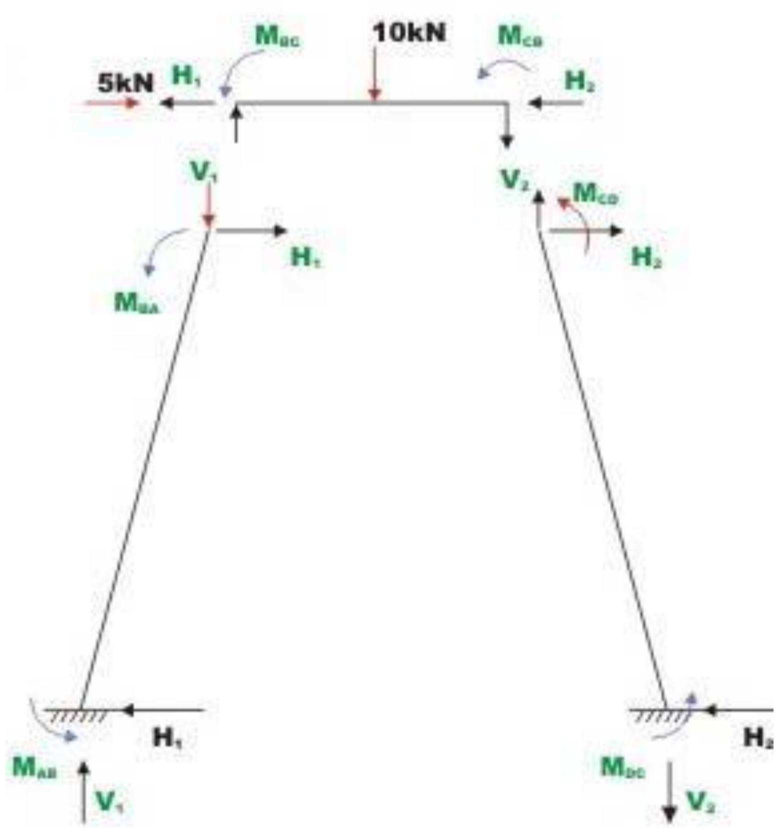


Fig.17.5c Free- body diagram

Shear equation for

Column *AB*

$$5H_1 - M_{AB} - M_{BA} + (1)V_1 = 0 \tag{5}$$

Column *CD*

$$5H_2 - M_{CD} - M_{DC} + (1)V_2 = 0 \tag{6}$$

Beam *BC*

$$\sum M_C = 0 \qquad 2V_1 - M_{BC} - M_{CB} - 10 = 0 \tag{7}$$

$$\sum F_X = 0 \qquad H_1 + H_2 = 5 \qquad (8)$$

$$\sum F_Y = 0 \qquad V_1 - V_2 - 10 = 0 \qquad (9)$$

From equation (7), $V_1 = \dfrac{M_{BC} + M_{CB} + 10}{2}$

From equation (8), $H_1 = 5 - H_2$

From equation (9), $V_2 = V_1 - 10 = \dfrac{M_{BC} + M_{CB} + 10}{2} - 10$

Substituting the values of V_1 , H_1 and V_2 in equations (5) and (6),

$$60 - 10H_2 - 2M_{AB} - 2M_{BA} + M_{BC} + M_{CB} = 0 \qquad (10)$$

$$-10 + 10H_2 - 2M_{CD} - 2M_{DC} + M_{BC} + M_{CB} = 0 \qquad (11)$$

Eliminating H_2 in equation (10) and (11),

$$M_{AB} + M_{BA} + M_{CD} + M_{DC} - M_{BC} - M_{CB} = 25 \qquad (12)$$

Substituting the values of M_{AB} , M_{BA} , M_{CD} , M_{DC} in (12) we get the required third equation. Thus,

$$0.784EI\theta_B + 0.471EI + 1.568EI\theta_B + 0.471EI + 1.568EI\theta_C + 0.471EI +$$
$$0.784EI\theta_C + 0.471EI \text{ -(} 2.5 + 2EI\theta_B + EI\theta_C - 0.6EI \text{)-}$$
$$(- 2.5 + EI\theta_B + 2EI\theta_C - 0.6EI) = 25$$

Simplifying,

$$- 0.648EI\theta_C - 0.648EI\theta_B + 3.084EI = 25 \qquad (13)$$

Solving simultaneously equations (3) (4) and (13), yields

$$EI\theta_B = -0.741 ; \qquad EI\theta_C = 1.205 \quad \text{and} \quad EI = 8.204 .$$

Substituting the values of $EI\theta_B$, $EI\theta_C$ and EI in the slope-deflection equation (3), one could calculate beam end moments. Thus,

$$M_{AB} = 3.28 \text{ kN.m}$$

UNIT-IV
MOMENT DISTRIBUTION METHOD

Objectives

After reading this chapter the student will be able to

1. Calculate stiffness factors and distribution factors for various members in a continuous beam.
2. Define unbalanced moment at a rigid joint.
3. Compute distribution moment and carry-over moment.
4. Derive expressions for distribution moment, carry-over moments.
5. Analyse continuous beam by the moment-distribution method.

Introduction

In the previous lesson we discussed the slope-deflection method. In slope-deflection analysis, the unknown displacements (rotations and translations) are related to the applied loading on the structure. The slope-deflection method results in a set of simultaneous equations of unknown displacements. The number of simultaneous equations will be equal to the number of unknowns to be evaluated. Thus one needs to solve these simultaneous equations to obtain displacements and beam end moments. Today, simultaneous equations could be solved very easily using a computer. Before the advent of electronic computing, this really posed a problem as the number of equations in the case of multistory building is quite large. The moment-distribution method proposed by Hardy Cross in 1932, actually solves these equations by the method of successive approximations. In this method, the results may be obtained to any desired degree of accuracy. Until recently, the moment-distribution method was very popular among engineers. It is very simple and is being used even today for preliminary analysis of small structures. It is still being taught in the classroom for the simplicity and physical insight it gives to the analyst even though stiffness method is being used more and more. Had the computers not emerged on the scene, the moment-distribution method could have turned out to be a very popular method. In this lesson, first moment-distribution method is developed for continuous beams with unyielding supports.

Basic Concepts

In moment-distribution method, counterclockwise beam end moments are taken as positive. The counterclockwise beam end moments produce clockwise moments on the joint Consider a continuous beam *ABCD* as shown in Fig.18.1a. In this beam, ends *A* and *D* are fixed and hence, $\theta_A = \theta_D = 0$ .Thus, the deformation of this beam is completely defined by rotations θ_B and θ_C at joints *B* and *C* respectively. The required equation to evaluate θ_B and θ_C is obtained by considering equilibrium of joints *B* and *C*. Hence,

$$\sum M_B = 0 \quad \Rightarrow M_{BA} + M_{BC} = 0 \tag{18.1a}$$

$$\sum M_C = 0 \quad \Rightarrow M_{CB} + M_{CD} = 0 \tag{18.1b}$$

According to slope-deflection equation, the beam end moments are written as

$$M_{BA} = M_{BA}^F + \frac{2EI_{AB}}{L_{AB}}(2\theta_B)$$

$\frac{4EI_{AB}}{L_{AB}}$ is known as stiffness factor for the beam *AB* and it is denoted by k_{AB}. M_{BA}^F is the fixed end moment at joint *B* of beam *AB* when joint *B* is fixed. Thus,

$$M_{BA} = M_{BA}^F + K_{AB}\theta_B$$

$$M_{BC} = M_{BC}^F + K_{BC}\left(\theta_B + \frac{\theta_C}{2}\right)$$

$$M_{CB} = M_{CB}^F + K_{CB}\left(\theta_C + \frac{\theta_B}{2}\right)$$

$$M_{CD} = M_{CD}^F + K_{CD}\theta_C \tag{18.2}$$

In Fig.18.1b, the counterclockwise beam-end moments M_{BA} and M_{BC} produce a clockwise moment M_B on the joint as shown in Fig.18.1b. To start with, in moment-distribution method, it is assumed that joints are locked i.e. joints are prevented from rotating. In such a case (vide Fig.18.1b),
$\theta_B = \theta_C = 0$, and hence

$$\begin{aligned} M_{BA} &= M_{BA}^F \\ M_{BC} &= M_{BC}^F \\ M_{CB} &= M_{CB}^F \\ M_{CD} &= M_{CD}^F \end{aligned} \tag{18.3}$$

Since joints *B* and *C* are artificially held locked, the resultant moment at joints *B* and *C* will not be equal to zero. This moment is denoted by M_B and is known as the unbalanced moment.

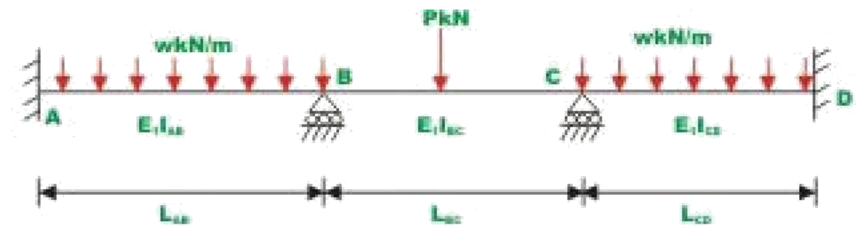


Fig. 18.1a Continuous Beam

Fig. 18.1b Continuous beam with fixed joints.

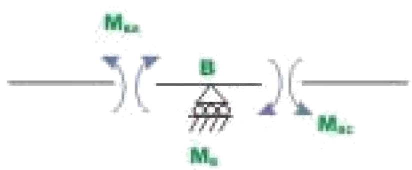


Fig. 18.1c Free - body diagram of joints B

Thus,

$$M_B = M_{BA}^F + M_{BC}^F$$

In reality joints are not locked. Joints *B* and *C* do rotate under external loads. When the joint *B* is unlocked, it will rotate under the action of unbalanced moment M_B . Let the joint B rotate by an angle θ_{B1} , under the action of M_B . This will deform the structure as shown in Fig.18.1d and introduces distributed moment M_{BA}^d, M_{BC}^d in the span BA and BC respectively as shown in the figure. The unknown distributed moments are assumed to be positive and hence act in counterclockwise direction. The unbalanced moment is the algebraic sum of the fixed end moments and act on the joint in the clockwise direction. The unbalanced moment restores the equilibrium of the joint B. Thus,

$$\sum M_B = 0, \quad M_{BA}^d + M_{BC}^d + M_B = 0 \qquad (18.4)$$

The distributed moments are related to the rotation θ_{B1} by the slope-deflection equation.

$$M_{BA}^{d} = K_{BA}\theta_{B1}$$

$$M_{BC}^{d} = K_{BC}\theta_{B1} \qquad (18.5)$$

Substituting equation (18.5) in (18.4), yields

$$\theta_{B1}\left(K_{BA} + K_{BC}\right) = -M_{B}$$

$$\theta_{B1} = -\frac{M_{B}}{K_{BA} + K_{BC}}$$

In general,

$$\theta_{B1} = -\frac{M_{B}}{\sum K} \qquad (18.6)$$

where summation is taken over all the members meeting at that particular joint. Substituting the value of θ_{B1} in equation (18.5), distributed moments are calculated. Thus,

$$M_{BA}^{d} = -\frac{K_{BA}}{\sum K} M_{B}$$

$$M_{BC}^{d} = -\frac{K_{BC}}{\sum K} M_{B} \qquad (18.7)$$

The ratio $\frac{K_{BA}}{\sum K}$ is known as the distribution factor and is represented by DF_{BA}. Thus,

$$M_{BA}^{d} = -DF_{BA}.\, M_{B}$$

$$M_{BC}^{d} = -DF_{BC}.\, M_{B} \qquad (18.8)$$

The distribution moments developed in a member meeting at *B*, when the joint *B* is unlocked and allowed to rotate under the action of unbalanced moment M_{B} is equal to a distribution factor times the unbalanced moment with its sign reversed.

As the joint *B* rotates under the action of the unbalanced moment, beam end moments are developed at ends of members meeting at that joint and are known as distributed moments. As the joint *B* rotates, it bends the beam and beam end moments at the far ends (i.e. at *A* and *C*) are developed. They are known as carry over moments. Now consider the beam *BC* of continuous beam *ABCD*.

When the joint *B* is unlocked, joint *C* is locked .The joint *B* rotates by θ_{B1} under the action of unbalanced moment M_B (vide Fig. 18.1e). Now from slope-deflection equations

$$M_{BC}^{d} = K_{BC}\theta_B$$

$$M_{BC} = \frac{1}{2} K_{BC}\theta_B$$

$$M_{CB} = \frac{1}{2} M_{BC}^{d} \qquad (18.9)$$

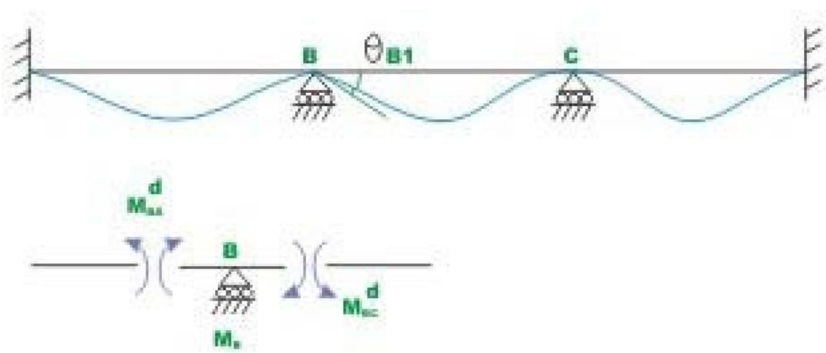


Fig. 18.1d Joint B is unlocked keeping C locked.

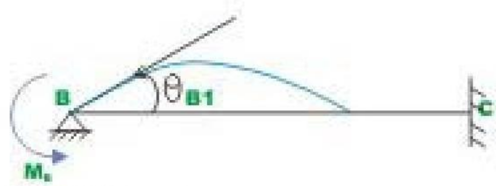


Fig.18.1e Carry - over moment

The carry over moment is one half of the distributed moment and has the same sign. With the above discussion, we are in a position to apply moment-distribution method to statically indeterminate beam. Few problems are solved here to illustrate the procedure. Carefully go through the first problem, wherein the moment-distribution method is explained in detail.

Example

A continuous prismatic beam *ABC* (see Fig.18.2a) of constant moment of inertia is carrying a uniformly distributed load of 2 kN/m in addition to a concentrated load of 10 kN. Draw bending moment diagram. Assume that supports are unyielding.

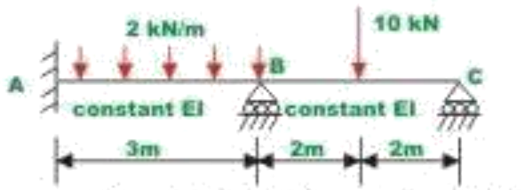


Fig. 18.2a Example 18.1

Solution

Assuming that supports B and C are locked, calculate fixed end moments developed in the beam due to externally applied load. Note that counterclockwise moments are taken as positive.

$$M_{AB}^{F} = \frac{wL_{AB}^{2}}{12} = \frac{2\times 9}{12} = 1.5 \text{ kN.m}$$

$$M_{BA}^{F} = -\frac{wL_{AB}^{2}}{12} = -\frac{2\times 9}{12} = -1.5 \text{ kN.m}$$

$$M_{BC}^{F} = \frac{Pab^{2}}{L_{BC}^{2}} = \frac{10\times 2\times 4}{16} = 5 \text{ kN.m}$$

$$M_{CB}^{F} = -\frac{Pa^{2}b}{L_{BC}^{2}} = -\frac{10\times 2\times 4}{16} = -5 \text{ kN.m} \qquad (1)$$

Before we start analyzing the beam by moment-distribution method, it is required to calculate stiffness and distribution factors.

$$K_{BA} = \frac{4EI}{3}$$

$$K_{BC} = \frac{4EI}{4}$$

At B: $\sum K = 2.333EI$

$$DF_{BA} = \frac{1.333EI}{2.333EI} = 0.571$$

$$DF_{BC} = \frac{EI}{2.333EI} = 0.429$$

At C: $\sum K = EI$

$DF_{CB} = 1.0$

Note that distribution factor is dimensionless. The sum of distribution factor at a joint, except when it is fixed is always equal to one. The distribution moments are developed only when the joints rotate under the action of unbalanced moment. In the case of fixed joint, it does not rotate and hence no distribution moments are developed and consequently distribution factor is equal to zero.
In Fig.18.2b the fixed end moments and distribution factors are shown on a working diagram. In this diagram *B* and *C* are assumed to be locked.

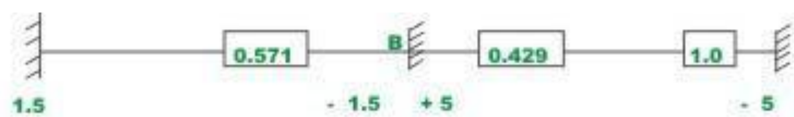


Fig. 18.2b

Now unlock the joint *C*. Note that joint *C* starts rotating under the unbalanced moment of 5 kN.m (counterclockwise) till a moment of -5 kN.m is developed (clockwise) at the joint. This in turn develops a beam end moment of +5 kN.m (M_{CB}). This is the distributed moment and thus restores equilibrium. Now joint C is relocked and a line is drawn below +5 kN.m to indicate equilibrium. When joint *C* rotates, a carry over moment of +2.5 kN.m is developed at the B end of member *BC*.These are shown in Fig.18.2c.

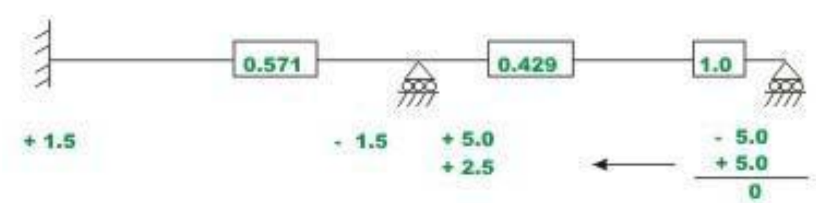


Fig. 18.2c

When joint B is unlocked, it will rotate under an unbalanced moment equal to algebraic sum of the fixed end moments(+5.0 and -1.5 kN.m) and a carry over moment of +2.5 kN.m till distributed moments are developed to restore equilibrium. The unbalanced moment is 6 kN.m. Now the distributed moments M_{BC} and M_{BA} are obtained by multiplying the unbalanced moment with the corresponding distribution factors and reversing the sign. Thus,

$M_{BC} = -2.574$ kN.m and $M_{BA} = -3.426$ kN.m. These distributed moments restore the equilibrium of joint *B*. Lock the joint *B*. This is shown in Fig.18.2d along with the carry over moments.

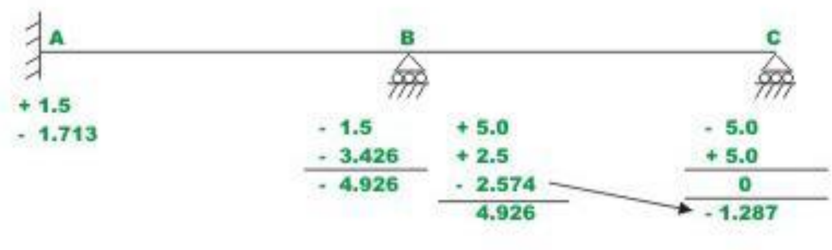


Fig. 18.2d

Now, it is seen that joint *B* is balanced. However joint *C* is not balanced due to the carry over moment -1.287 kN.m that is developed when the joint *B* is allowed to rotate. The whole procedure of locking and unlocking the joints *C* and *B* successively has to be continued till both joints B and C are balanced simultaneously. The complete procedure is shown in Fig.18.2e.

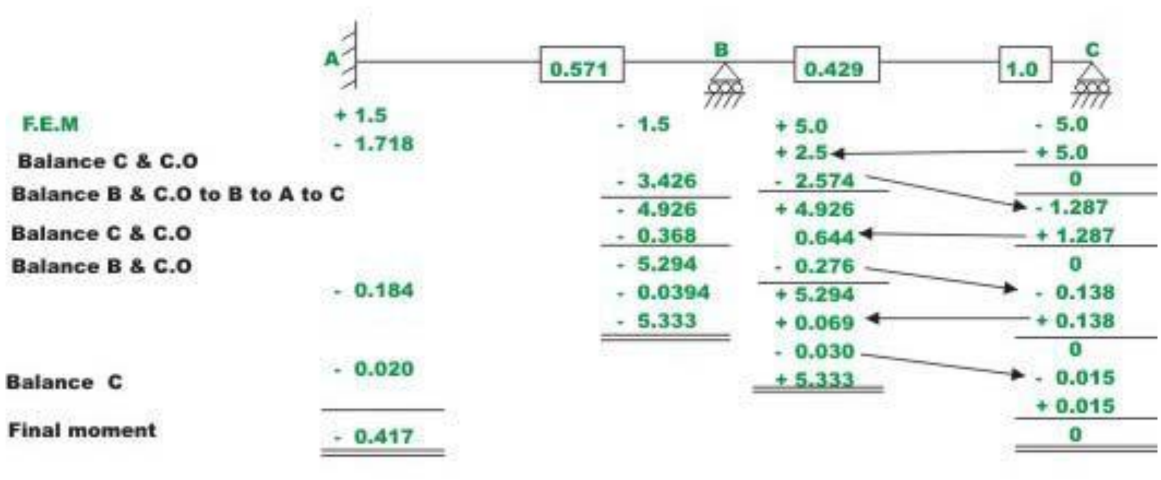


Fig. 18.2e Moment - distribution method : Computation

The iteration procedure is terminated when the change in beam end moments is less than say 1%. In the above problem the convergence may be improved if we leave the hinged end *C* unlocked after the first cycle. This will be discussed in the next section. In such a case the stiffness of beam BC gets modified. The above calculations can also be done conveniently in a tabular form as shown in Table 18.1. However the above working method is preferred in this course.

Table 18.1 Moment-distribution for continuous beam ABC

Joint	A	B		C
Member	AB	BA	BC	CB
Stiffness	1.333EI	1.333EI	EI	EI
Distribution factor		0.571	0.429	1.0
FEM in kN.m	+1.5	-1.5	+5.0	-5.0
Balance joints C ,B and C.O.	-1.713	-3.426	+2.5 -2.579	+5.0 0
		-4.926	+4.926	-1.287
Balance C and C.O.			+0.644	1.287
Balance B and C.O.		-0.368	-0.276	-0.138
Balance C	-0.184	-5.294	+5.294	0.138
C.O.			+0.069	0
Balance B and C.O.	-0.02	-0.039	-0.030	-0.015
Balance C				+0.015
Balanced moments in kN.m	-0.417	-5.333	+5.333	0

Modified stiffness factor when the far end is hinged

As mentioned in the previous example, alternate unlocking and locking at the hinged joint slows down the convergence of moment-distribution method. At the hinged end the moment is zero and hence we could allow the hinged joint C in the previous example to rotate freely after unlocking it first time. This necessitates certain changes in the stiffness parameters. Now consider beam *ABC* as shown in Fig.18.2a. Now if joint *C* is left unlocked then the stiffness of member *BC* changes. When joint *B* is unlocked, it will rotate by θ_{B1} under the action of unbalanced moment M_B .The support *C* will also rotate by θ_{C1} as it is free to rotate. However, moment $M_{CB} = 0$. Thus

$$M_{CB} = K_{BC}\theta_C + \frac{K_{BC}}{2}\theta_B \qquad (18.7)$$

But, $M_{CB} = 0$

$$\Rightarrow \theta_C = -\frac{\theta_B}{2} \qquad (18.8)$$

Now,

$$M_{BC} = K_{BC}\theta_B + \frac{K_{BC}}{2}\theta_C \qquad (18.9)$$

Substituting the value of θ_C in eqn. (18.9),

$$M_{BC} = K_{BC}\theta_B - \frac{K_{BC}}{4}\theta_B = \frac{3}{4}K_{BC}\theta_B \qquad (18.10)$$

$$M_{BC} = K_{BC}^{R}\theta_B \qquad (18.11)$$

The K_{BC}^{R} is known as the reduced stiffness factor and is equal to $\frac{3}{4}K_{BC}$ .Accordingly distribution factors also get modified. It must be noted that there is no carry over to joint *C* as it was left unlocked.

Example 2

Solve the previous example by making the necessary modification for hinged end *C*.

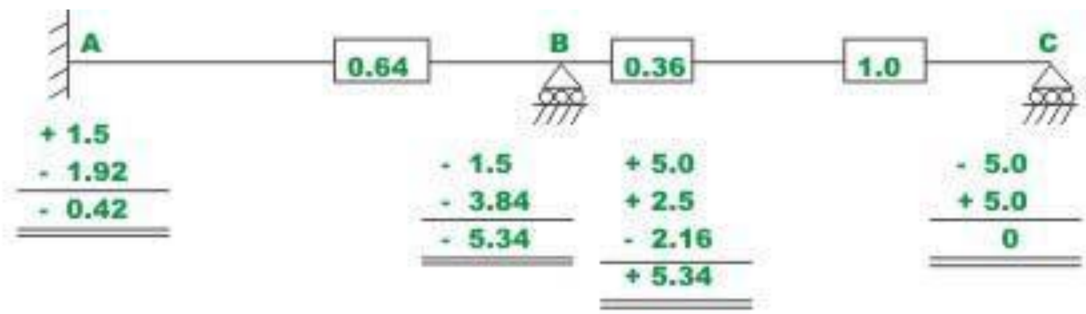


Fig. 18.3 Example 18.2

Fixed end moments are the same. Now calculate stiffness and distribution factors.

$$K_{BA} = 1.333EI, K_{BC} = \frac{3}{4}EI = 0.75EI$$

Joint B: $\sum K = 2.083,\quad D_{BA}^{F} = 0.64\ ,\ D_{BC}^{F} = 0.36$

Joint C: $\sum K = 0.75EI,\quad D_{CB}^{F} = 1.0$

All the calculations are shown in Fig.18.3a

Please note that the same results as obtained in the previous example are obtained here in only one cycle. All joints are in equilibrium when they are unlocked. Hence we could stop moment-distribution iteration, as there is no unbalanced moment anywhere.

Example 3

Draw the bending moment diagram for the continuous beam *ABCD* loaded as shown in Fig.18.4a.The relative moment of inertia of each span of the beam is also shown in the figure.

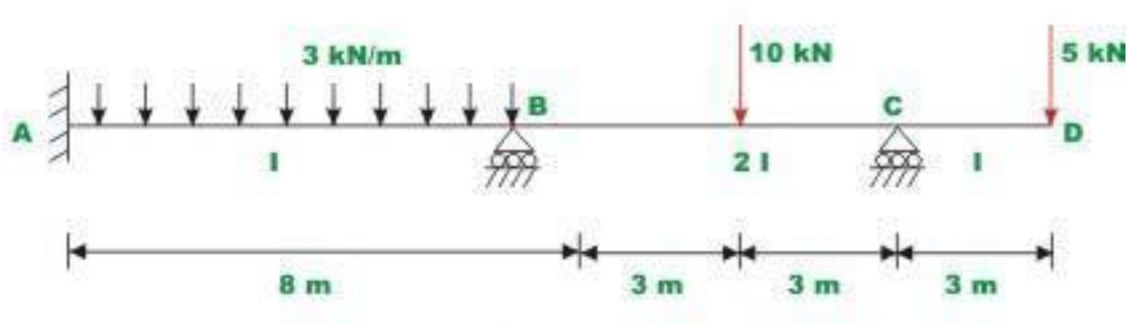


Fig. 18.4a Example 18.3

Solution

Note that joint *C* is hinged and hence stiffness factor *BC* gets modified. Assuming that the supports are locked, calculate fixed end moments. They are

$$M_{AB}^{F} = 16 \text{ kN.m}$$

$$M_{BA}^{F} = -16 \text{ kN.m}$$

$$M_{BC}^{F} = 7.5 \text{ kN.m}$$

$$M_{CB}^{F} = -7.5 \text{ kN.m} \text{ , and}$$

$$M_{CD}^{F} = 15 \text{ kN.m}$$

In the next step calculate stiffness and distribution factors

$$K_{BA} = \frac{4EI}{8}$$

$$K_{BC} = \frac{3}{4}\frac{8EI}{6}$$

At joint B:

$$\sum K = 0.5EI + 1.0EI = 1.5EI$$

$$D_{BA}^{F} = \frac{0.5EI}{1.5EI} = 0.333$$

$$D_{BC}^{F} = \frac{1.0EI}{1.5EI} = 0.667$$

At *C*:

$$\sum K = EI, \quad D_{CB}^{F} = 1.0$$

Now all the calculations are shown in Fig.18.4b

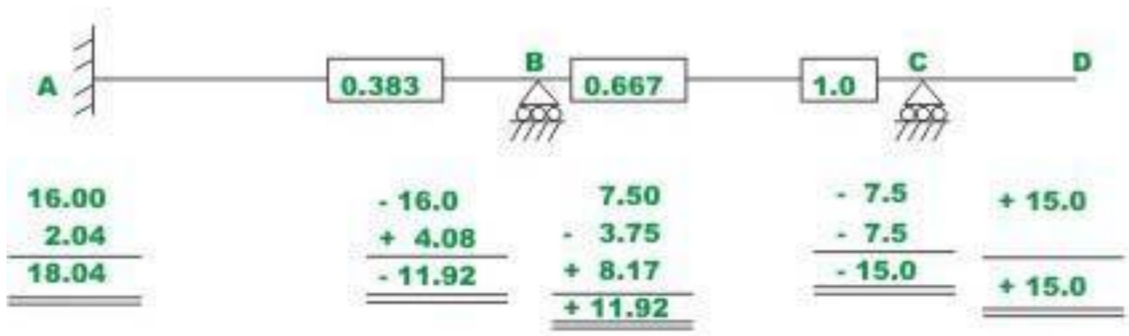


Fig. 18.4b Computation

This problem has also been solved by slope-deflection method (see example 14.2).The bending moment diagram is shown in Fig.18.4c.

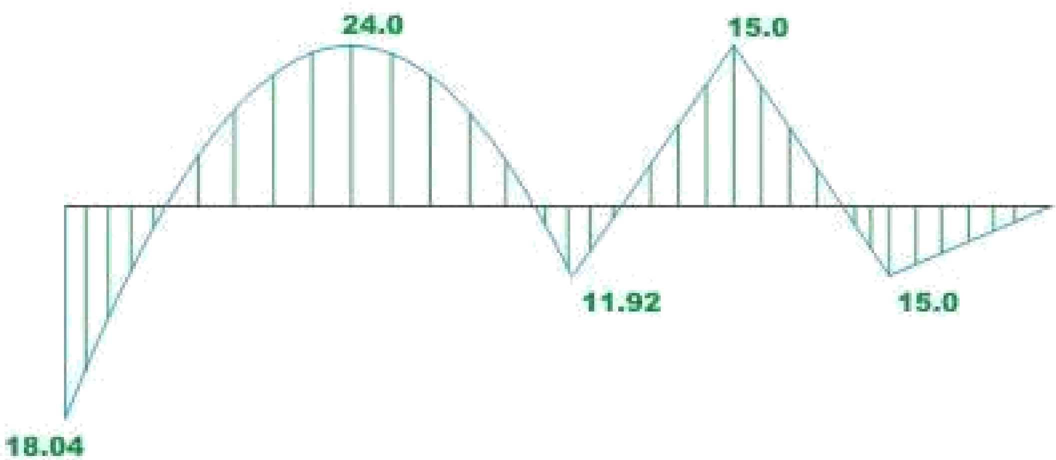


Fig. 18.4c Bending - moment diagram

Instructional Objectives

After reading this chapter the student will be able to

1. Solve continuous beam with support settlements by the moment-distribution method.
2. Compute reactions at the supports.
3. Draw bending moment and shear force diagrams.
4. Draw the deflected shape of the continuous beam.

Introduction

In the previous lesson, moment-distribution method was discussed in the context of statically indeterminate beams with unyielding supports. It is very well known that support may settle by unequal amount during the lifetime of the structure. Such support settlements induce fixed end moments in the beams so as to hold the end slopes of the members as zero (see Fig. 19.1).

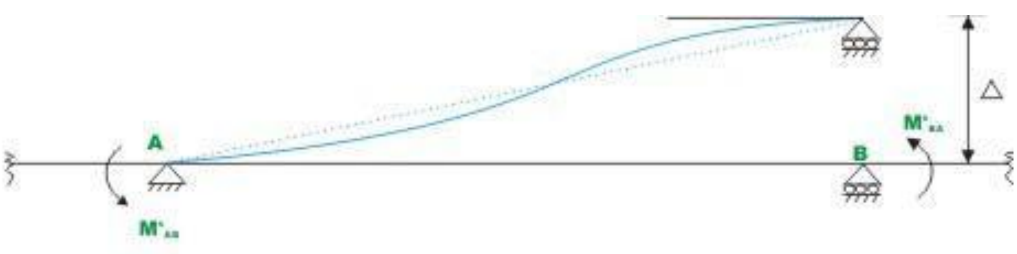


Fig . 19.1 Support settlement without ratation

In lesson 15, an expression (equation 15.5) for beam end moments were derived by superposing the end moments developed due to

1. Externally applied loads on beams
2. Due to displacements θ_A, θ_B and (settlements).

The required equations are,

$$M_{AB} = M_{AB}^{F} + \frac{2EI_{AB}}{L_{AB}} \quad 2\theta_A + \theta_B - \frac{3}{L_{AB}} \qquad (19.1a)$$

$$M_{BA} = M_{BA}^F + \frac{2EI_{AB}}{L_{AB}} 2\theta_B + \theta_A - \frac{3}{L_{AB}} \qquad (19.1b)$$

This may be written as,

$$M_{AB} = M_{AB}^F + 2K_{AB}\left[2\theta_A + \theta_B\right] + M_{AB}^S \qquad (19.2a)$$

$$M_{BA} = M_{BA}^F + 2K_{AB}\left[2\theta_B + \theta_A\right] + M_{BA}^S \qquad (19.2b)$$

where $K_{AB} = \frac{EI_{AB}}{L_{AB}}$ is the stiffness factor for the beam *AB*. The coefficient 4 has been dropped since only relative values are required in calculating distribution factors.

M_{AB}^S

Note that $M_{AB}^S = M_{BA}^S = -\frac{6EI_{AB}}{L^2_{AB}}$ (19.3)

is the beam end moments due to support settlement and is negative (clockwise) for positive support settlements (upwards). In the moment-distribution method, the support moments M_{AB}^S and M_{BA}^S due to uneven support settlements are distributed in a similar manner as the fixed end moments, which were described in details in lesson 18.

It is important to follow consistent sign convention. Here counterclockwise beam taken as positive. The moment-distribution method as applied to statically indeterminate beams undergoing uneven support settlements is illustrated with a few examples.

Example 1

Calculate the support moments of the continuous beam ABC (Fig. 19.2a) having constant flexural rigidity EI throughout, due to vertical settlement of support B by 5mm. Assume $E = 200$ GPa ; and $I = 4\times10^{-4}\ \text{m}^4$.

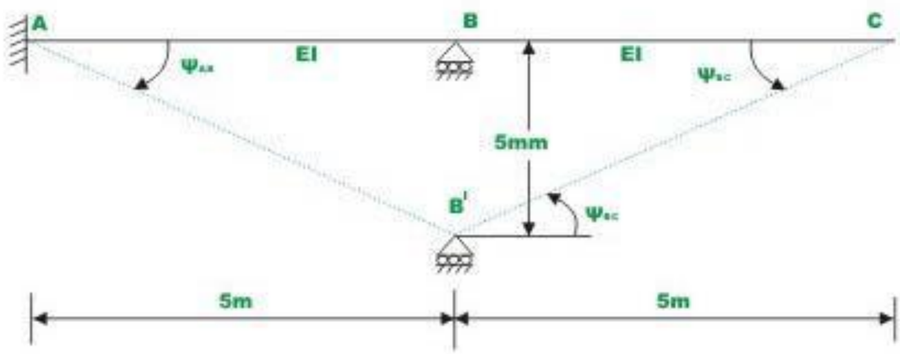


Fig . 19.2a Chord rotation due to support settlement (Example 19.1)

Solution

There is no load on the beam and hence fixed end moments are zero. However, fixed end moments are developed due to support settlement of B by 5mm. In the span AB , the chord rotates by ψ_{AB} in clockwise direction. Thus,

$$\psi_{AB} = -\frac{5\times10^{-3}}{5}$$

$$M^S_{AB} = M^S_{BA} = -\frac{6EI_{AB}}{L_{AB}}\psi_{AB} = -\frac{6\times200\times10^9\times4\times10^{-4}}{5}\left(-\frac{5\times10^{-3}}{5}\right)$$

$$= 96000 \text{ Nm} = 96 \text{ kNm.} \qquad (1)$$

In the span BC , the chord rotates by ψ_{BC} in the counterclockwise direction and hence taken as positive.

$$\psi_{BC} = \frac{5\times10^{-3}}{5}$$

$$M^{S}_{BC} = M^{S}_{CB} = -\frac{6EI_{BC}}{L_{BC}}\psi_{BC} = -\frac{6 \times 200 \times 10^{9} \times 4 \times 10^{-4}}{5}\frac{5 \times 10^{-3}}{5}$$

$$= -96000\ Nm = -96\ kNm. \qquad (2)$$

Now calculate stiffness and distribution factors.

$$K_{BA} = \frac{EI_{AB}}{L_{AB}} = 0.2EI \text{ and } K_{BC} = \frac{3}{4}\frac{EI_{BC}}{L_{BC}} = 0.15EI \qquad (3)$$

Note that, while calculating stiffness factor, the coefficient 4 has been dropped since only relative values are required in calculating the distribution factors. For span BC, reduced stiffness factor has been taken as support C is hinged.

At B:

$$\sum K = 0.35EI$$

$$DF_{BA} = \frac{0.2EI}{0.35EI} = 0.571$$

$$DF_{BC} = \frac{0.15EI}{0.35EI} = 0.429 \qquad (4)$$

At support C:

$$\sum K = 0.15EI\ ; \qquad DF_{CB} = 1.0\ .$$

Now joint moments are balanced as discussed previously by unlocking and locking each joint in succession and distributing the unbalanced moments till the joints have rotated to their final positions. The complete procedure is shown in Fig. 19.2b and also in Table 19.1.

Fig. 19.2b Computation

Table 19.1 Moment-distribution for continuous beam ABC

Joint	A	B		C
Member		BA	BC	CB
Stiffness factor		0.2EI	0.15EI	0.15EI
Distribution Factor		0.571	0.429	1.000
Fixed End Moments (kN.m)	96.000	96.000	-96.000	-96.000
Balance joint C and C.O. to B			48.00	96.000
Balance joint B and C.O. to A	-13,704	-27.408	-20.592	
Final Moments (kN.m)	82.296	68.592	-68.592	0.000

Note that there is no carry over to joint C as it was left unlocked.

Example 2

A continuous beam $ABCD$ is carrying uniformly distributed load $5\ kN/m$ as shown in Fig. 19.3a. Compute reactions and draw shear force and bending moment diagram due to following support settlements.

Support B, 0.005m vertically downwards Support C, .0100m vertically downwards. Assume $E = 200GPa$; $I = 1.35 \times 10^{-3}\ m^4$.

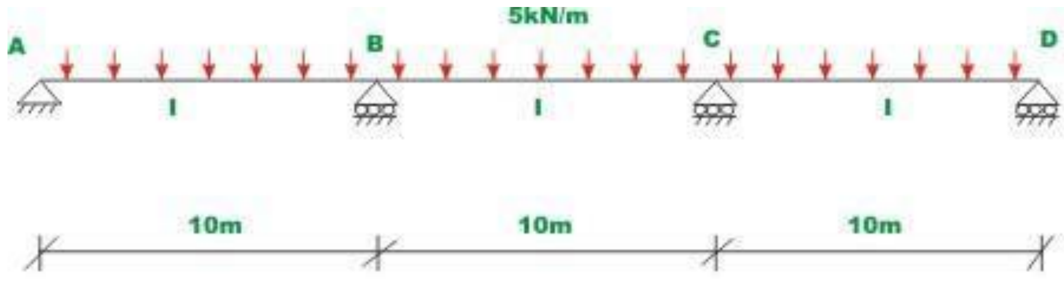


Fig .19.3a Continuous beam of Example 19.2

Solution:

Assume that supports A, B, C and D are locked and calculate fixed end moments due to externally applied load and support settlements. The fixed end beam moments due to externally applied loads are,

$$M_{AB}^{F} = \frac{5 \times 100}{12} = 41.67 \text{ kN.m}; \; M_{BA}^{F} = -41.67 \text{ kN.m}$$

$$M_{BC}^{F} = +41.67 \text{ kN.m}; \quad M_{BC}^{F} = -41.67 \text{ kN.m}$$

$$M_{CD}^{F} = +41.67 \text{ kN.m}; \quad M_{DC}^{F} = -41.67 \text{ kN.m} \qquad (1)$$

In the span AB , the chord joining joints A and B rotates in the clockwise direction as B moves vertical downwards with respect to A (see Fig. 19.3b).

$\psi_{AB} = -0.0005$ radians (negative as chord AB' rotates in the clockwise direction from its original position)

$\psi_{BC} = -0.0005$ radians

$\psi_{CD} = 0.001$ radians (positive as chord $C'D$ rotates in the counterclockwise direction).

Now the fixed end beam moments due to support settlements are,

$$M_{AB}^{S} = -\frac{6EI_{AB}}{L_{AB}} \psi_{AB} = -\frac{6 \times 200 \times 10^{9} \times 1.35 \times 10^{-3}}{10} (-0.0005)$$

$$= 81000 \text{ N.m} = 81.00 \text{ kN.m}$$

$$M_{BA}^{S} = 81.00 \text{ kN.m}$$

$$M_{BC}^{S} = M_{CB}^{S} = 81.00 \text{ kN.m}$$

$$M_{CD}^{S} = M_{DC}^{S} = -162.00 \text{ kN.m} \qquad (3)$$

In the next step, calculate stiffness and distribution factors. For span AB and CD modified stiffness factors are used as supports A and D are hinged. Stiffness factors are,

$$K_{BA} = \frac{3}{4}\frac{EI}{10} = 0.075EI \; ; \qquad K_{BC} = \frac{EI}{10} = 0.10EI$$

$$K_{CB} = \frac{EI}{10} = 0.10EI \; ; \qquad K_{CD} = \frac{3}{4}\frac{EI}{10} = 0.075EI \qquad (4)$$

At joint A : $\sum K = 0.075EI$; $DF_{AB} = 1.0$

At joint B : $\sum K = 0.175EI$; $DF_{BA} = 0.429$; $DF_{BC} = 0.571$

At joint C : $\sum K = 0.175EI$; $DF_{CB} = 0.571$; $DF_{CD} = 0.429$

At joint D : $\sum K = 0.075EI$; $DF_{DC} = 1.0$

The complete procedure of successively unlocking the joints, balancing them and locking them is shown in a working diagram in Fig.19.3c. In the first row, the distribution factors are entered. Then fixed end moments due to applied loads and support settlements are entered. In the first step, release joints A and D . The unbalanced moments at A and D are 122.67 kN.m, -203.67 kN.m respectively. Hence balancing moments at A and D are -122.67 kN.m, 203.67 kN.m respectively. (Note that we are dealing with beam end moments and not joint moments).

The joint moments are negative of the beam end moments. Further leave A and D unlocked as they are hinged joints. Now carry over moments -61.34 kN.m and 101.84 kN.m to joint B and C respectively. In the next cycle, balance joints B and C. The unbalanced moment at joint B is 100.66 kN.m. Hence balancing moment for beam BA is −43.19 (− 100.66 ×0.429) and for BC is −57.48 kN.m (-100.66 x 0.571). The balancing moment on BC gives a carry over moment of −26.74 kN.m to joint C. The whole procedure is shown in Fig. 19.3c and in Table 19.2. It must be noted that there is no carryover to joints A and D as they were left unlocked.

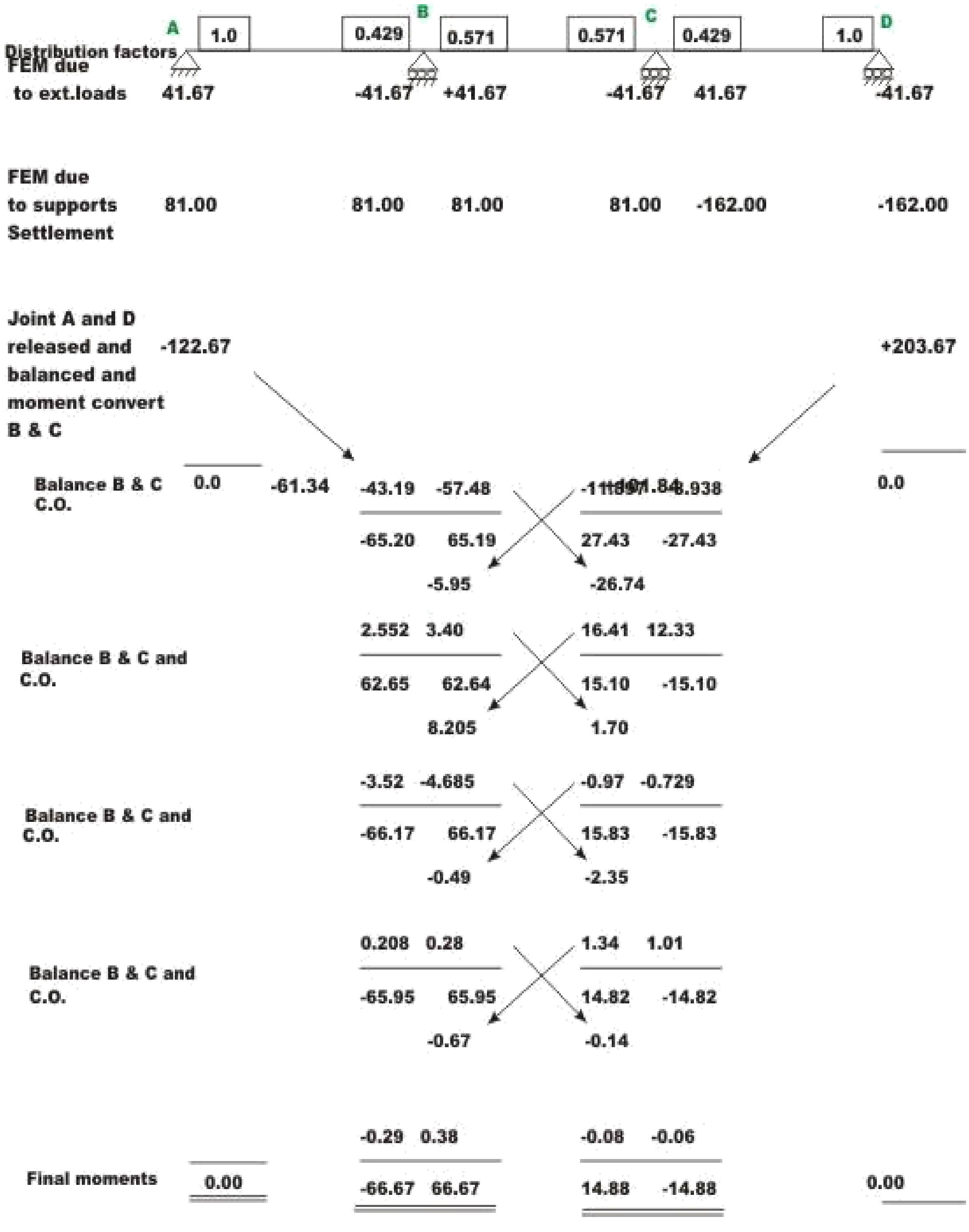


Fig. 19.3 © Computation

Table 19.2 Moment-distribution for continuous beam ABCD

Joint	A	B		C		D
Members	AB	BA	BC	CB	CD	DC
Stiffness factors	0.075 EI	0.075 EI	0.1 EI	0.1 EI	0.075 EI	0.075 EI
Distribution Factors	1.000	0.429	0.571	0.571	0.429	1.000
FEM due to externally applied loads	41.670	-41.670	41.670	-41.670	41.670	-41.670
FEM due to support settlements	81.000	81.000	81.000	81.000	-162.000	-162.000
Total	122.670	39.330	122.670	39.330	-120.330	-203.670
Balance A and D released	-122.670					203.670
Carry over		-61.335			101.835	
Balance B and C		-43.185	-57.480	-11.897	-8.94	
Carry over			-5.95	-26.740		
Balance B and C		2.552	3.40	16.410	12.33	
Carry over to B and C			8.21	1.70		
Balance B and C		-3.52	-4.69	-0.97	-0.73	
C.O. to B and C			-0.49	-2.33		
Balance B and C		0.21	0.28	1.34	1.01	
Carry over			0.67	0.14		
Balance B and C		-0.29	-0.38	-0.08	-0.06	
Final Moments	0.000	-66.67	66.67	14.88	-14.88	0.000

Example 3

Analyse the continuous beam ABC shown in Fig. 19.4a by moment-distribution method. The support B settles by $5mm$ below A and C. Assume EI to be constant for all members $E = 200GPa$; and $I = 8 \times 10^6\ mm^4$.

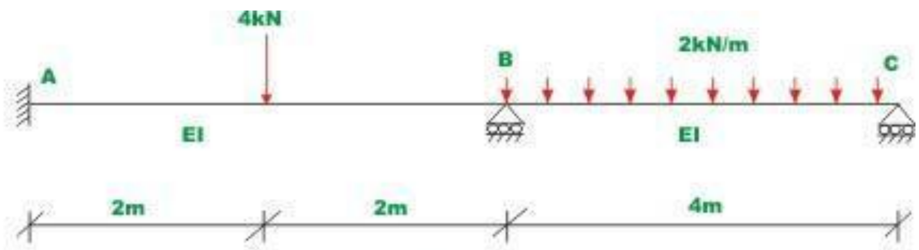


Fig. 19.4 (a) Example 19.4a

Solution:

Calculate fixed end beam moments due to externally applied loads assuming that support B and C are locked.

$$M_{AB}^F = +2\ kN.m; \qquad M_{BA}^F = -2\ kN.m$$
$$M_{BC}^F = +2.67\ kN.m; \qquad M_{CB}^F = -2.67\ kN.m \qquad (1)$$

In the next step calculate fixed end moments due to support settlements. In the span AB, the chord AB' rotates in the clockwise direction and in span BC, the chord $B'C$ rotates in the counterclockwise direction (Fig. 19.4b).

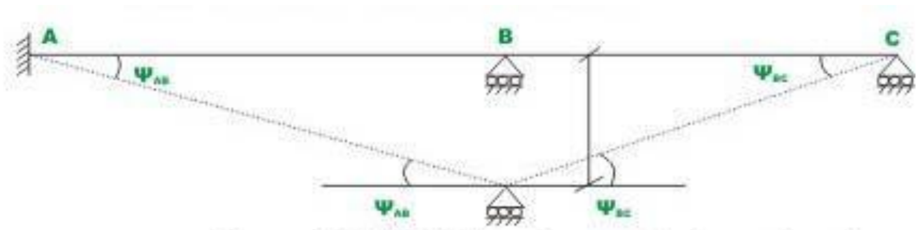


Fig. 19.4 (b) Member rotation due to support settlement

$$\psi_{AB} = -\frac{5\times10^{-3}}{4} = -1.25\times10^{-3}\ radians$$

$$\psi_{BC} = \frac{5\times10^{-3}}{4} = 1.25\times10^{-3}\ radians \qquad (2)$$

$$M_{AB}^{S} = M_{BA}^{S} = -\frac{6EI_{AB}}{L_{AB}}\psi_{AB} = -\frac{6\times200\times10^{9}\times8\times10^{-6}}{4} - \frac{5\times10^{-3}}{4}$$

$$= 3000\ Nm = 3\ kNm. \qquad (3)$$

$$M_{BC}^{S} = M_{CB}^{S} = -3.0 kN.m$$

In the next step, calculate stiffness and distribution factors.

$$K_{AB} = K_{BA} = 0.25EI$$

$$K_{BC} = \frac{3}{4}0.25EI = 0.1875EI \qquad (4)$$

At joint B : $\sum K = 0.4375EI$; $DF_{BA} = 0.571$; $DF_{BC} = 0.429$

At joint C : $\sum K = 0.1875EI$; $DF_{CB} = 1.0$

At fixed joint, the joint does not rotate and hence no distribution moments are developed and consequently distribution factor is equal to zero. The complete moment-distribution procedure is shown in Fig. 19.4c and Table 19.3. The diagram is self explanatory. In this particular case results are obtained in two cycles. In the first cycle joint C is balanced and carry over moment is taken to joint B . In the next cycle , joint B is balanced and carry over moment is taken to joint A . The bending moment diagram is shown in fig. 19.4d.

Table 19.3 Moment-distribution for continuous beam ABC

Joints	A	B		C
Member	AB	BA	BC	CB
Stiffness factor	0.25 EI	0.25 EI	0.1875 EI	0.1875 EI
Distribution Factor		0.571	0.429	1.000
Fixed End Moments due to applied loads (kN.m)	2.000	-2.000	2.667	-2.667
Fixed End Moments due to support settlements (kN.m)	3.000	3.000	-3.000	-3.000
Total	5.000	1.000	-0.333	-5.667
Balance joint C and C.O.			2.835	5.667
Total	5.000	1.000	2.502	0.000
Balance joint B and C.O. to A	-1.00	-2.000	-1.502	
Final Moments (kN.m)	4.000	-1.000	1.000	0.000

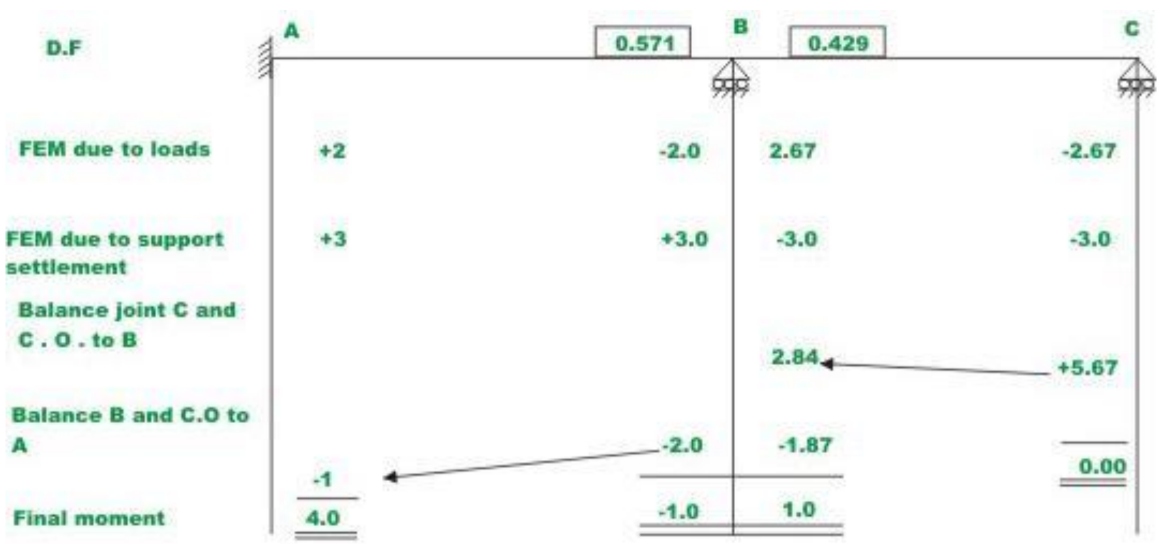


Fig. 19.4 (c) Computation

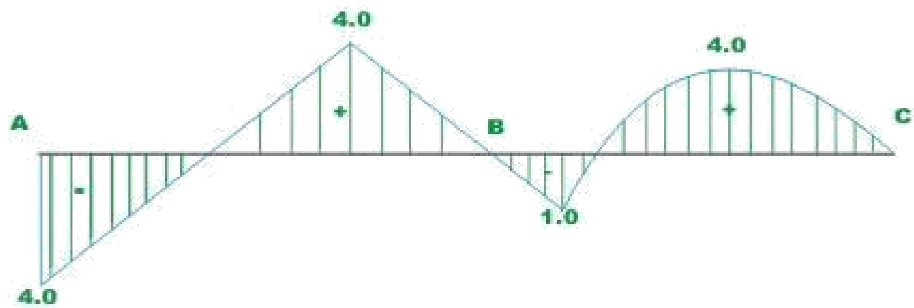


FIG. 19.4 (d) B.M.D

Instructional Objectives

After reading this chapter the student will be able to

1. Solve plane frame restrained against sidesway by the moment-distribution method.
2. Compute reactions at the supports.
3. Draw bending moment and shear force diagrams.
4. Draw the deflected shape of the plane frame.

Introduction

In this lesson, the statically indeterminate rigid frames properly restrained against sidesway are analysed using moment-distribution method. Analysis of rigid frames by moment-distribution method is very similar to that of continuous beams described in lesson 18. As pointed out earlier, in the case of continuous beams, at a joint only two members meet, where as in case of rigid frames two or more than two members meet at a joint. At such joints (for example joint C in Fig. 20.1) where more than two members meet, the unbalanced moment at the beginning of each cycle is the algebraic sum of fixed end beam moments (in the first cycle) or the carry over moments (in the subsequent cycles) of the beam meeting at C . The unbalanced moment is distributed to members CB, CD and CE according to their distribution factors. Few examples are solved to explain procedure. The moment-distribution method is carried out on a working diagram.

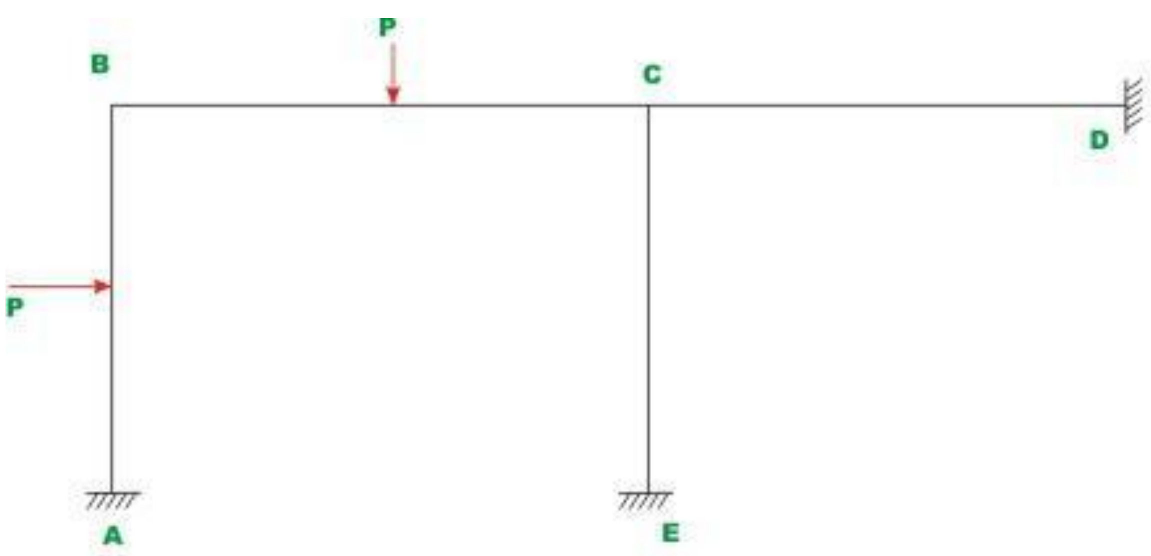


Fig. 20.1 Plane frame

Example 1

Calculate reactions and beam end moments for the rigid frame shown in Fig. 20.2a. Draw bending moment diagram for the frame. Assume EI to be constant for all the members.

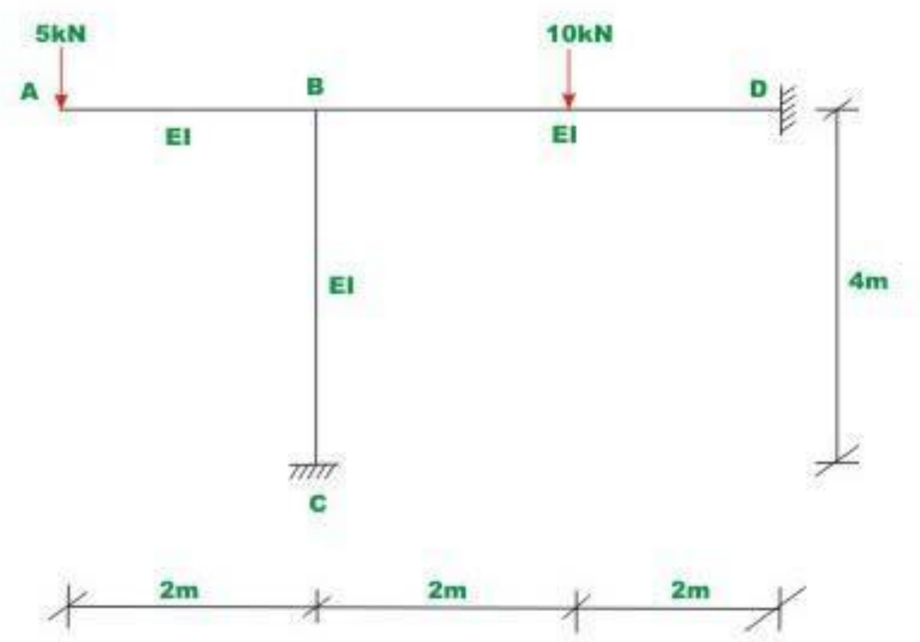


Fig. 20.2a Rigid plane frame of Example 20.1

Solution

In the first step, calculate fixed end moments.

$$M_{BD}^{F} = 5.0\ kN.m$$

$$M_{DB}^{F} = -5.0\ kN.m \qquad (1)$$

$$M_{BC}^{F} = 0.0\ kN.m$$

$$M_{CB}^{F} = 0.0\ kN.m$$

Also, the fixed end moment acting at B on BA is clockwise.

$$M_{BA}^{F} = -10.0\ kN.m$$

In the next step calculate stiffness and distribution factors.

$$K_{BD} = \frac{EI}{4} = 0.25EI \quad \text{and} \quad K_{BC} = \frac{EI}{4} = 0.25EI$$

At joint B :

$$\sum K = 0.50EI$$

$$DF_{BD} = \frac{0.25EI}{0.5EI} = 0.5 \; ; \qquad DF_{BC} = 0.5 \qquad (2)$$

All the calculations are shown in Fig. 20.2b. Please note that cantilever member does not have any restraining effect on the joint B from rotation. In addition its stiffness factor is zero. Hence unbalanced moment is distributed between members BC and BD only.

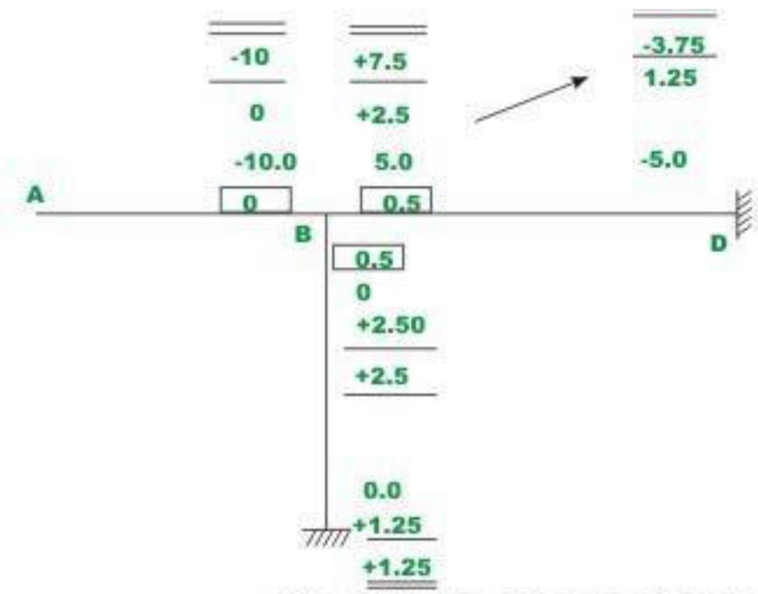


Fig. 20.2b Moment distribution

In this problem the moment-distribution method is completed in only one cycle, as equilibrium of only one joint needs to be considered. In other words, there is only one equation that needs to be solved for the unknown θ_B in this problem.

This problem has already been solved by slop- deflection method wherein reactions are computed from equations of statics. The free body diagram of each member of the frame with external load and beam end moments are again reproduced here in Fig. 20.2c for easy reference. The bending moment diagram is shown in Fig. 20.2d.

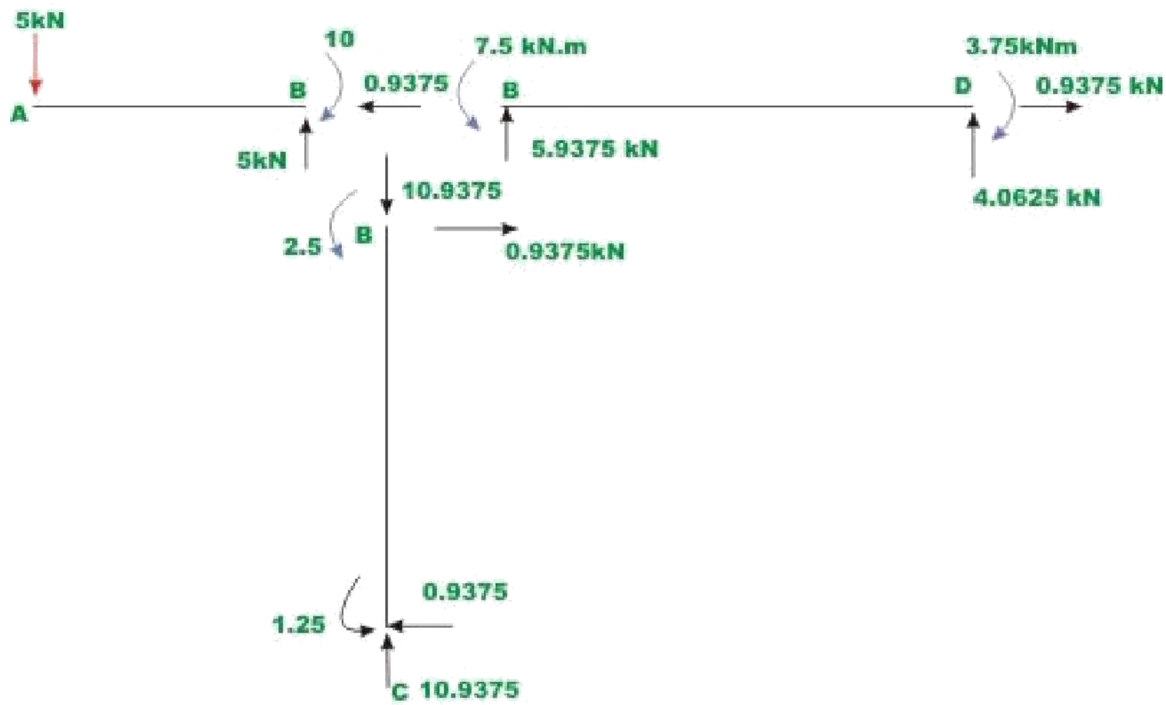


Fig. 20.2c Reactions

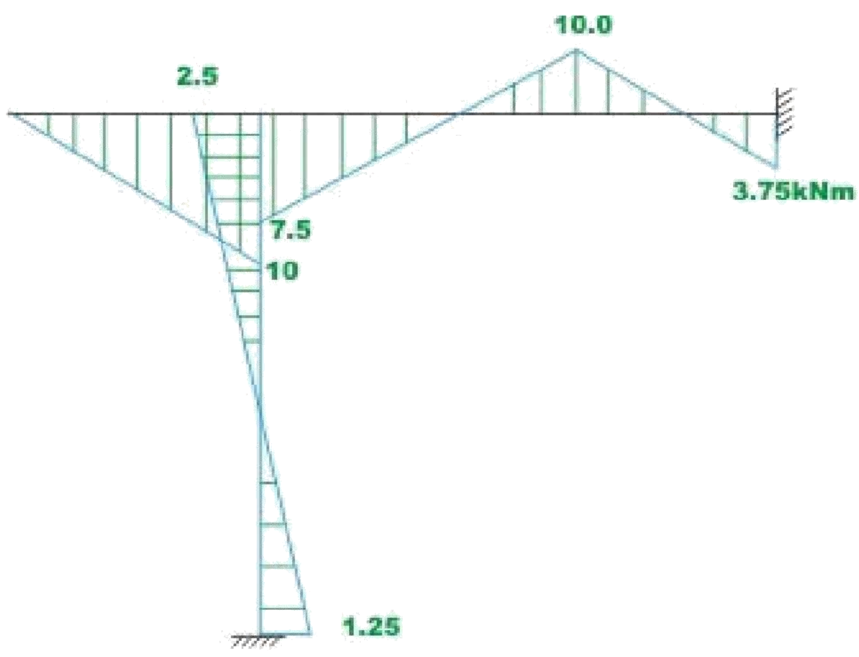


Fig. 20.2 (d) Bending moment diagram

Example 2

Analyse the rigid frame shown in Fig. 20.3a by moment-distribution method. Moment of inertia of different members are shown in the diagram.

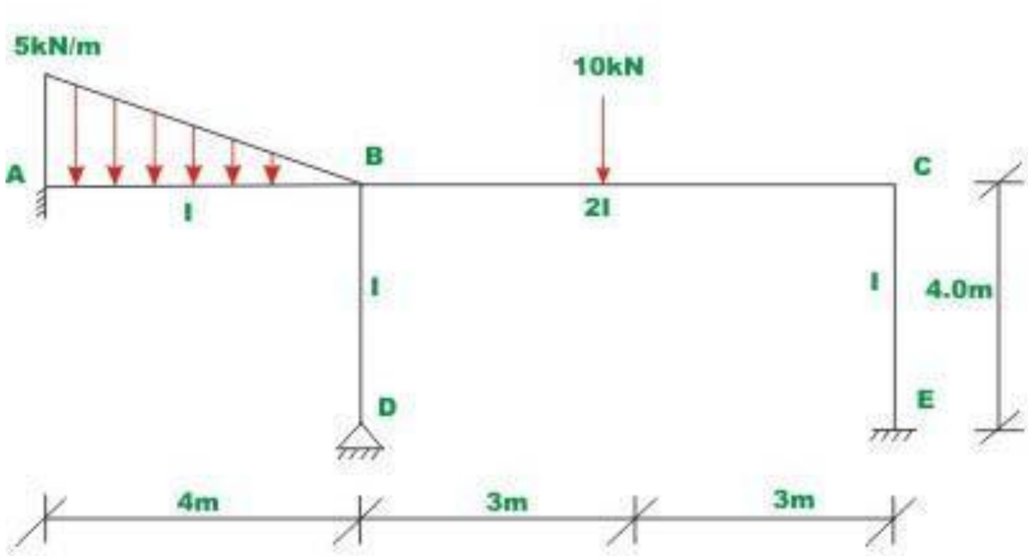


Fig. 20.3 (a) Example 20.2

Solution:

Calculate fixed end moments by locking the joints *A*, *B*, *C*, *D* and *E*

$$M_{AB}^F = \frac{5 \times 4^2}{20} = 4.0 \text{ kN.m}$$

$$M_{BA}^F = -2.667 \text{ kN.m}$$

$$M_{BC}^F = 7.5 \text{ kN.m}$$

$$M_{CB}^F = -7.5 \text{ kN.m}$$

$$M_{BD}^F = M_{DB}^F = M_{CE}^F = M_{EC}^F = 0 \qquad (1)$$

The frame is restrained against sidesway. In the next step calculate stiffness and distribution factors.

$$K_{BA} = 0.25EI \text{ and } K_{BC} = \frac{2EI}{6} = 0.333EI$$

$$K_{BD} = \frac{3}{4}\frac{EI}{4} = 0.1875EI \; ; \qquad K_{CE} = 0.25EI \qquad (2)$$

At joint B :

$$\sum K = K_{BA} + K_{BC} + K_{BD}$$

$$= 0.7705EI$$

$$DF_{BA} = 0.325 \; ; \qquad DF_{BC} = 0.432$$

$$DF_{BD} = 0.243 \qquad (3)$$

At joint C :

$$\sum K = 0.583EI$$

$$DF_{CB} = 0.571 \; ; \qquad DF_{CD} = 0.429$$

In Fig. 20.3b, the complete procedure is shown on a working diagram. The moment-distribution method is started from joint C . When joint C is unlocked, it will rotate under the action of unbalanced moment of 7.5 kN.m . Hence the 7.5 kN.m is distributed among members CB and CE according to their distribution factors. Now joint C is balanced. To indicate that the joint C is balanced a horizontal line is drawn. This balancing moment in turn developed moments $+2.141 \text{ kN.m}$ at BC and $+1.61 \text{ kN.m}$ at EC . Now unlock joint B . The joint B is unbalanced and the unbalanced moment is $-(7.5 + 2.141 - 2.67) = -6.971 \text{ kN.m}$. This moment is distributed among three members meeting at B in proportion to their distribution factors. Also there is no carry over to joint D from beam end moment BD as it was left unlocked. For member BD , modified stiffness factor is used as the end D is hinged.

Example 3

Analyse the rigid frame shown in Fig. 20.4a by moment-distribution method. Draw bending moment diagram for the rigid frame. The flexural rigidities of the members are shown in the figure.

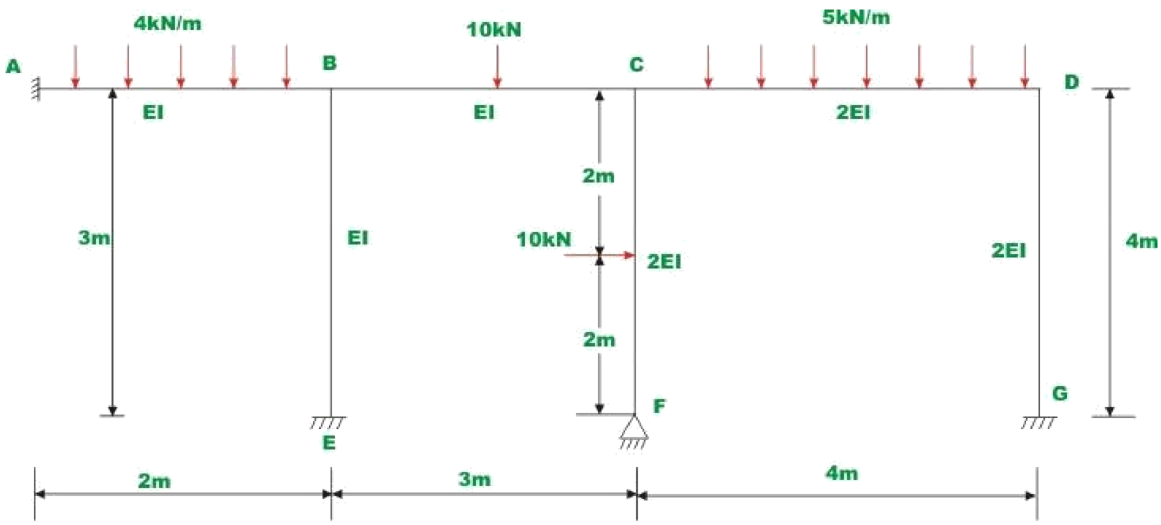


Fig. 20.4a Example 20.3

Solution:

Assuming that the joints are locked, calculate fixed end moments.

$$M_{AB}^{F} = 1.333 \text{ kN.m} \; ; M_{BA}^{F} = -1.333 \text{ kN.m}$$

$$M_{BC}^{F} = 4.444 \text{ kN.m}; \; M_{CB}^{F} = -2.222 \text{ kN.m}$$

$$M_{CD}^{F} = 6.667 \text{ kN.m} \; ; M_{DC}^{F} = -6.667 \text{ kN.m}$$

$$M_{BE}^{F} = 0.0 \text{ kN.m} \; ; M_{EB}^{F} = 0.0 \text{ kN.m}$$

$$M_{CF}^{F} = 5.0 \text{ kN.m} \; ; M_{FC}^{F} = -5.0 \text{ kN.m} \qquad (1)$$

The frame is restrained against sidesway. Calculate stiffness and distribution factors.

$$K_{BA} = 0.5EI \; ; \qquad K_{BC} = 0.333EI \; ; \qquad K_{BE} = 0.333EI$$

$$K_{CB} = 0.333EI \; ; \qquad K_{CD} = 0.5EI \; ; \qquad K_{CF} = \frac{3}{4}\frac{2EI}{4} = 0.375EI$$

$$K_{DC} = 0.5EI \; ; \qquad K_{DG} = 0.5EI \qquad (2)$$

Joint B **:**

$$\sum K = 0.5EI + 0.333EI + 0.333EI = 1.166EI$$

$DF_{BA} = 0.428$; $DF_{BC} = 0.286$

$DF_{BE} = 0.286$

Joint C :

$$\sum K = 0.333EI + 0.5EI + 0.375EI = 1.208EI$$

$DF_{CB} = 0.276$; $DF_{CD} = 0.414$

$DF_{CF} = 0.31$

Joint D :

$$\sum K = 1.0EI$$

$DF_{DC} = 0.50$; $DF_{DG} = 0.50$ (3)

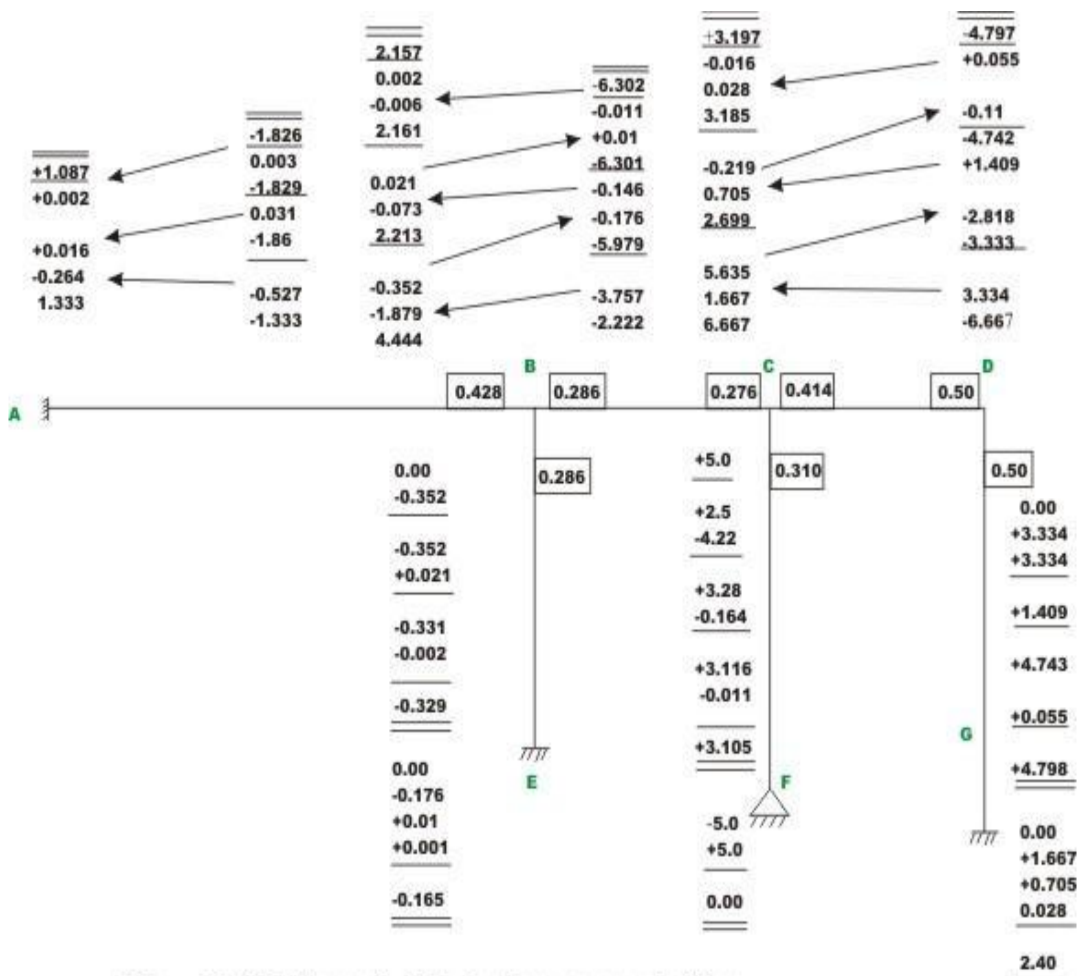


Fig. 20.4b Moment distribution computation

The complete moment-distribution method is shown in Fig. 20.4b. The moment-distribution is stopped after three cycles. The moment-distribution is started by releasing and balancing joint D . This is repeated for joints C and B respectively in that order. After balancing joint F , it is left unlocked throughout as it is a hinged joint. After balancing each joint a horizontal line is drawn to indicate that joint has been balanced and locked. When moment-distribution method is finally stopped all joints except fixed joints will be left unlocked.

Introduction

In the previous lesson, rigid frames restrained against sidesway are analyzed using moment-distribution method. It has been pointed in lesson 17, that frames which are unsymmetrical or frames which are loaded unsymmetrically usually get displaced either to the right or to the left. In other words, in such frames apart from evaluating joint rotations, one also needs to evaluate joint translations (sidesway). For example in frame shown in Fig 21.1, the loading is symmetrical but the geometry of frame is unsymmetrical and hence sidesway needs to be considered in the analysis. The number of unknowns is this case are: joint rotations θ_B and θ_C and member rotation ψ . Joint B and C get translated by the same amount as axial deformations are not considered and hence only one independent member rotation need to be considered. The procedure to analyze rigid frames undergoing lateral displacement using moment-distribution method is explained in section 21.2 using an example.

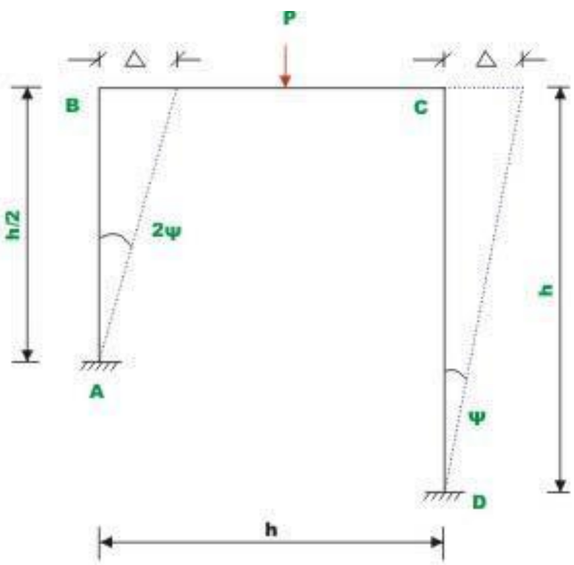


Fig 21.1 Rigid frame

Procedure

A special procedure is required to analyze frames with sidesway using moment-distribution method. In the first step, identify the number of independent rotations (ψ) in the structure. The procedure to calculate independent rotations is explained in lesson 22. For analyzing frames with sidesway, the method of superposition is used. The structure shown in Fig. 21.2a is expressed as the sum of two systems: Fig. 21.2b and Fig. 21.2c. The systems shown in figures 21.2b and 21.2c are analyzed separately and superposed to obtain the final answer. In system 21.2b, sidesway is prevented by artificial support at C. Apply all the external loads on frame shown in Fig. 21.2b. Since for the frame, sidesway is prevented, moment-distribution method as discussed in the previous lesson is applied and beam end moments are calculated. Let M_{AB}', M_{BA}', M_{BC}', M_{CB}', M_{CD}' and M_{DC}' be the balanced moments obtained by distributing fixed end moments due to applied loads while allowing only joint rotations (θ_B and θ_C) and preventing sidesway.

Now, calculate reactions H_{A1} and H_{D1} (ref. Fig 21.3a).they are ,

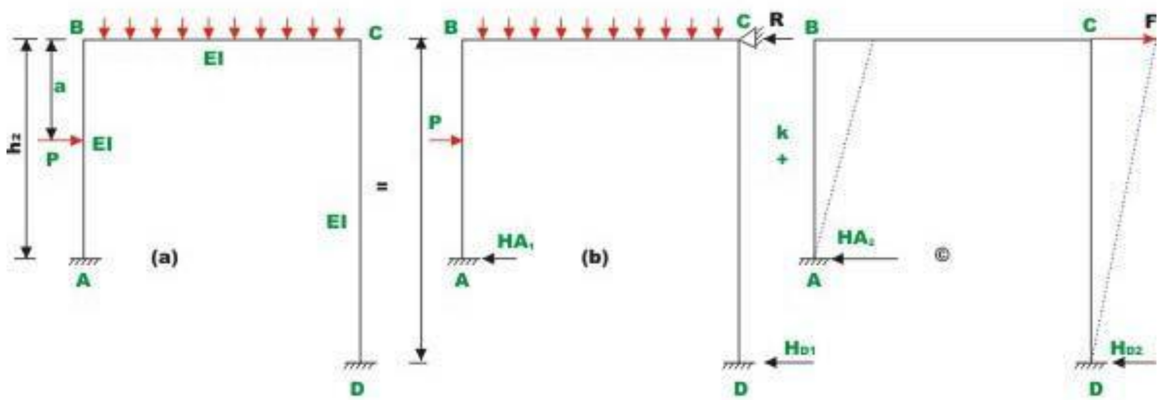


Fig 21.2 Frame with sidesway

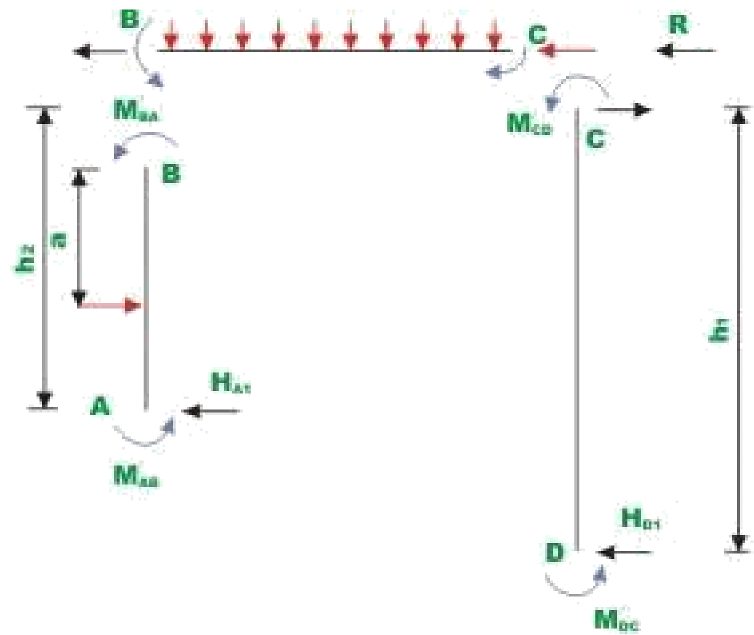


Fig.21.3a Free body diagram

$$H_{A1} = \frac{M'_{AB} + M'_{BA}}{h_2} + \frac{Pa}{h_2}$$

$$H_{D1} = \frac{M'_{CD} + M'_{DC}}{h_1} \qquad (21.1)$$

again, $$R = P - (H_{A1} + H_{D1}) \qquad (21.2)$$

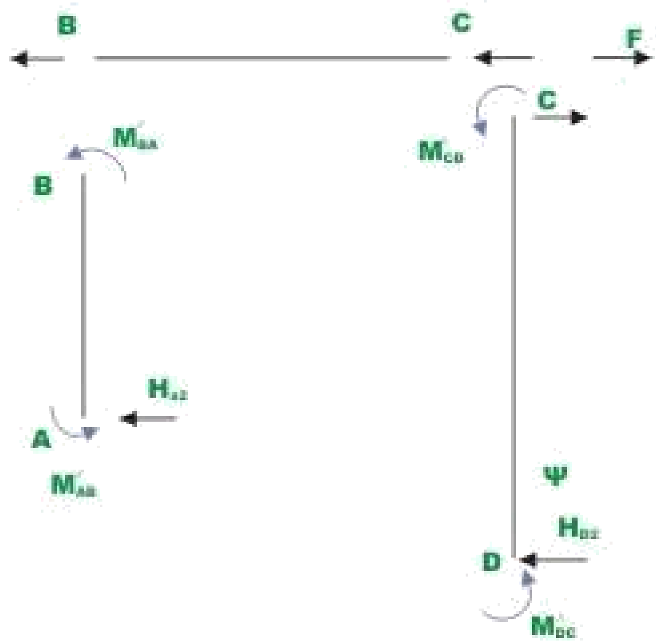


Fig.21.3b Free body diagram of frame

In Fig 21.2c apply a horizontal force F in the opposite direction of R . Now $k\,F = R$, then the superposition of beam end moments of system (b) and k times (c) gives the results for the original structure. However, there is no way one could analyze the frame for horizontal force F , by moment-distribution method as sway comes in to picture. Instead of applying F , apply arbitrary known displacement / sidesway ' as shown in the figure. Calculate the fixed end beam moments in the column AB and CD for the imposed horizontal displacement. Since joint displacement is known beforehand, one could use moment-distribution method to analyse this frame. In this case, member rotations ψ are related to joint translation which is known. Let $M_{AB}^{"}, M_{BA}^{"}, M_{BC}^{"}, M_{CB}^{"}, M_{CD}^{"}$ and $M_{DC}^{"}$ are the balanced moment obtained by distributing the fixed end moments due to assumed sidesway ' at joints B and C . Now, from statics calculate horizontal force F due to arbitrary sidesway ' .

$$H_{A2} = \frac{M''_{AB} + M''_{BA}}{h_2}$$

$$H_{D2} = \frac{M''_{CD} + M''_{DC}}{h_1} \qquad (21.3)$$

$$F = (H_{A2} + H_{D2}) \qquad (21.4)$$

In Fig 21.2, by method of superposition

$kF = R$ or $k = R / F$

Substituting the values of R and F from equations (21.2) and (21.4),

$$k = \frac{P - (H_{A1} + H_{D1})}{(H_{A2} + H_{D2})} \qquad (21.5)$$

Now substituting the values of H_{A1}, H_{A2}, H_{D1} and H_{D2} in 21.5,

$$k = \frac{P - \frac{M'_{AB} + M'_{BA}}{h_2} + \frac{Pa}{h_2} + \frac{M'_{CD} + M'_{DC}}{h_1}}{\frac{M''_{AB} + M''_{BA}}{h_2} + \frac{M''_{CD} + M''_{DC}}{h_1}} \qquad (21.6)$$

Hence, beam end moment in the original structure is obtained as,

$$M_{original} = M_{system(b)} + kM_{system(c)}$$

If there is more than one independent member rotation, then the above procedure needs to be modified and is discussed in the next lesson.

Example 1

Analyse the rigid frame shown in Fig 21.4a. Assume EI to be constant for all members. Also sketch elastic curve.

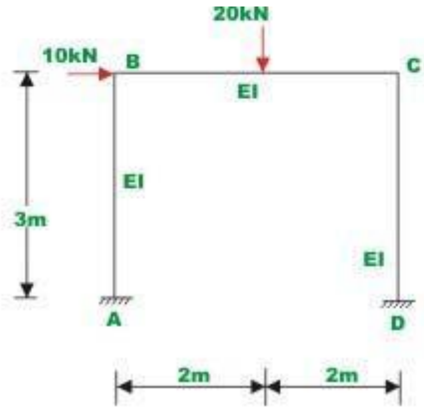


Fig. 21.4a Rigid frame of Example 21.1

Solution

In the given problem, joint C can also rotate and also translate by an unknown amount . This problem has to be solved in two steps. In the first step, evaluate the beam-end moment by preventing the sidesway.

In the second step calculate beam end moments by moment-distribution method for known translation (see Fig 21.4b). By appropriately superposing the two results, the beam end moment of the original structure is obtained.

a) Calculate stiffness and distribution factors

$$K_{BA} = 0.333EI \; ; \; K_{BC} = 0.25EI \; ;$$

$$K_{CB} = 0.25EI \; ; \; K_{CD} = 0.333EI$$

Joint B : $\sum K = 0.583EI$

$$DF_{BA} = 0.571; \; DF_{BC} = 0.429$$

Joint C : $\sum K = 0.583EI$

$$DF_{CB} = 0.429 \; ; DF_{CD} = 0.571 \; . \qquad (1)$$

b) Calculate fixed end moment due to applied loading.

$$M_{AB}^{F} = 0 \; ; \qquad M_{BA}^{F} = 0 \text{ kN.m}$$

$$M_{BC}^{F} = +10 \text{ kN.m} \; ; \quad M_{CB}^{F} = -10 \text{ kN.m}$$

$$M_{CD}^{F} = 0 \text{ kN.m} \quad ; \quad M_{DC}^{F} = 0 \text{ kN.m} \; . \qquad (2)$$

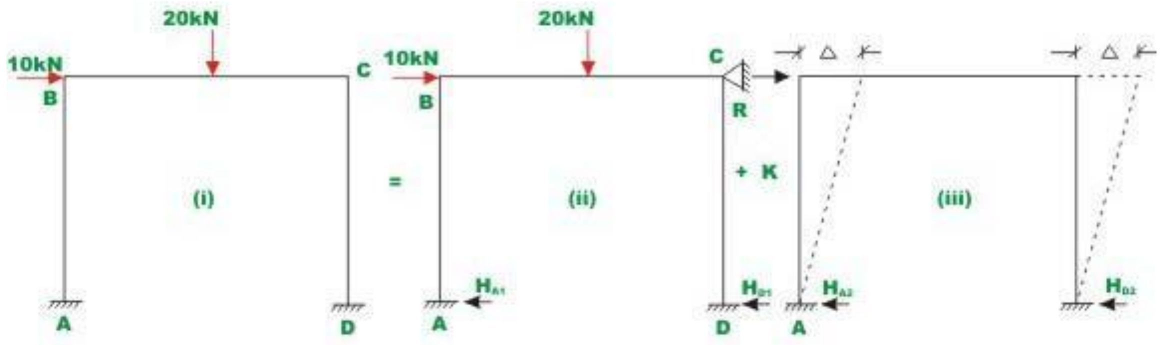


Fig. 21.4b Frame with side - sway

Now the frame is prevented from sidesway by providing a support at C as shown in Fig 21.4b (ii). The moment-distribution for this frame is shown in Fig 21.4c. Let M'_{AB}, M'_{BA}, M'_{CD} and M'_{DC} be the balanced end moments. Now calculate horizontal reactions at A and D from equations of statics.

$$H_{A1} = \frac{M'_{AB} + M'_{BA}}{3}$$

$$= \frac{-3.635 + 7.268}{3}$$

$$= -3.635 \; KN \; (\rightarrow) \; .$$

$$H_{D1} = \frac{3.636 - 17.269}{3} = 3.635 \text{ kN}(\leftarrow) \; .$$

$$R = 10 - (-3.635 + 3.635) = -10 \text{ kN}(\rightarrow) \qquad (3)$$

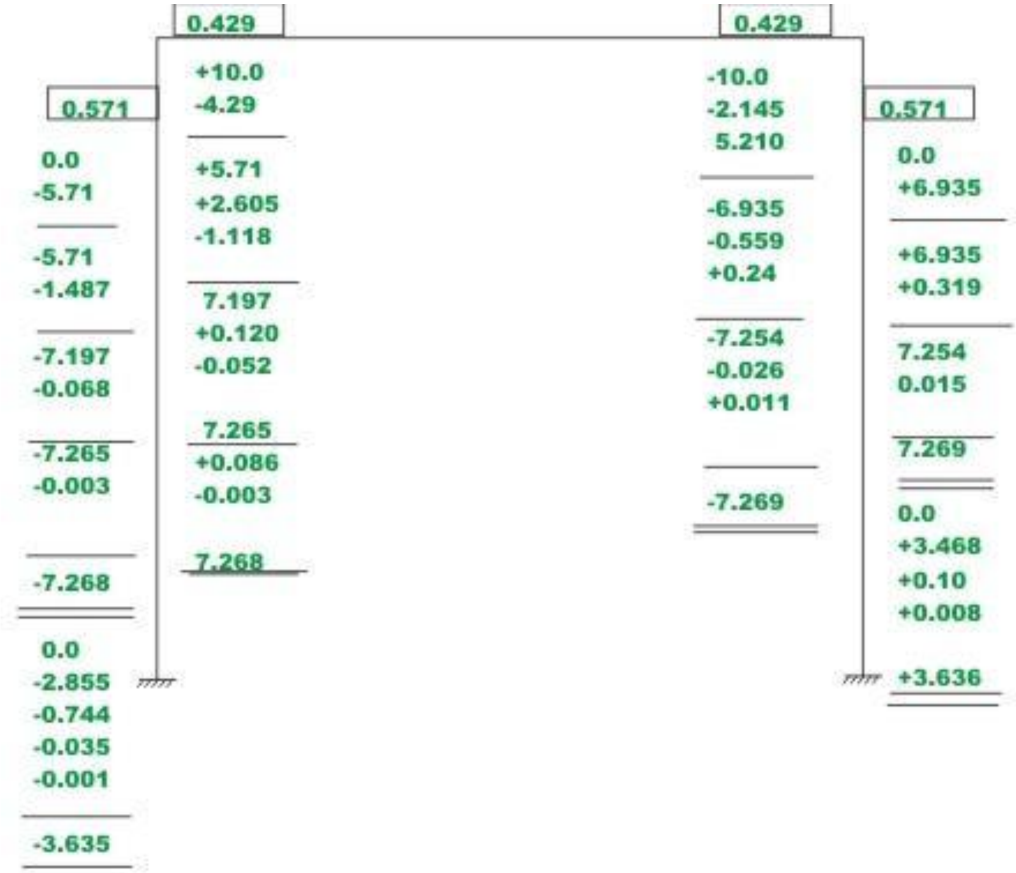


Fig. 21.4c Moment distribution with sidesway prevented

d) Moment-distribution for arbitrary known sidesway ' .

Since ' is arbitrary, Choose any convenient value. Let $' = \frac{150}{EI}$ Now calculate fixed end beam moments for this arbitrary sidesway.

$$M_{AB}^{F} = -\frac{6EI\psi}{L} = -\frac{6\,EI}{3} \times \left(-\frac{150}{3EI}\right) = 100 \text{ kN.m}$$

$$M_{BA}^{F} = 100 \text{ kN.m}$$

$$M_{CD}^{F} = M_{DC}^{F} = +100 \text{ kN.m} \qquad (4)$$

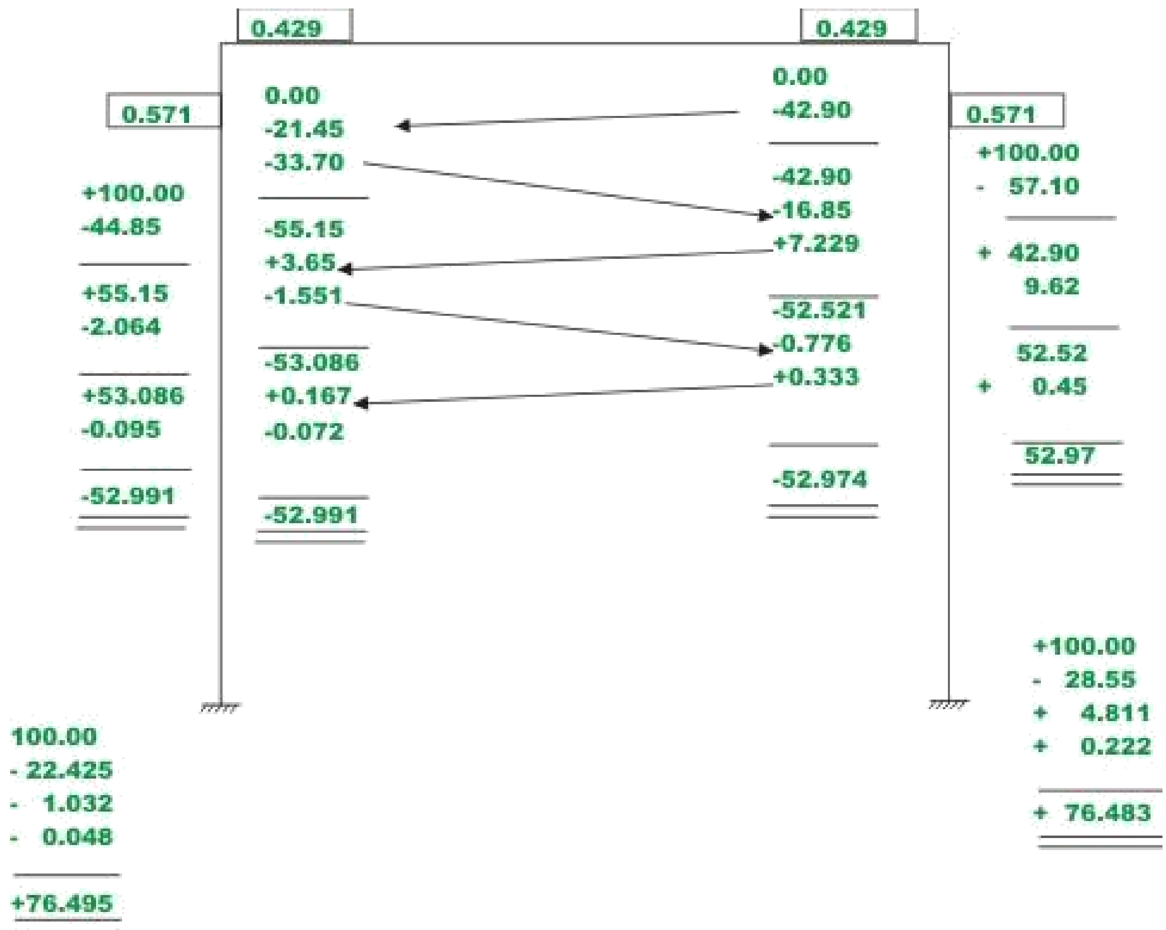


Fig. 21.4d Moment distribution for sidesway

The moment-distribution for this case is shown in Fig 24.4d. Now calculate horizontal reactions H_{A2} and H_{D2}.

$$H_{A2} = \frac{52.98 + 76.48}{3} = 43.15 \text{ kN}(\leftarrow)$$

$$H_{D2} = \frac{52.97 + 76.49}{3} = 43.15 \text{ kN}(\leftarrow)$$

$$F = -86.30 \text{ kN}(\rightarrow)$$

Let k be a factor by which the solution of case (iii) needs to be multiplied. Now actual moments in the frame is obtained by superposing the solution (ii) on the solution obtained by multiplying case (iii) by k . Thus kF cancel out the holding force R such that final result is for the frame without holding force.

Thus, $k F = R$.

$$k = \frac{-10}{-86.13} = 0.1161 \qquad (5)$$

Now the actual end moments in the frame are,

$$M_{AB} = M'_{AB} + k M''_{AB}$$

$$M_{AB} = -3.635 + 0.1161(+76.48) = +5.244 \text{ kN.m}$$

$$M_{BA} = -7.268 + 0.1161(+52.98) = -1.117 \text{ kN.m}$$

$$M_{BC} = +7.268 + 0.1161(-52.98) = +1.117 \text{ kN.m}$$

$$M_{CB} = -7.269 + 0.1161(-52.97) = -13.419 \text{ kN.m}$$

$$M_{CD} = +7.268 + 0.1161(+52.97) = +13.418 \text{ kN.m}$$

$$M_{DC} = +3.636 + 0.1161(+76.49) = +12.517 \text{ kN.m}$$

The actual sway is computed as,

$$= k' = 0.1161 \times \frac{150}{EI}$$

$$= \frac{17.415}{EI}$$

The joint rotations can be calculated using slope-deflection equations.

$$M_{AB} = M_{AB}^{F} + \frac{2EI}{L}\left[2\theta_A + \theta_B - 3\psi_{AB}\right] \qquad \text{where } \psi_{AB} = -\frac{\Delta}{L}$$

$$M_{BA} = M_{BA}^{F} + \frac{2EI}{L}\left[2\theta_B + \theta_A - 3\psi_{AB}\right]$$

In the above equation, except θ_A and θ_B all other quantities are known. Solving for θ_A and θ_B ,

$$\theta_A = 0 \;; \qquad \theta_B = \frac{-9.55}{EI}.$$

The elastic curve is shown in Fig. 21.4e.

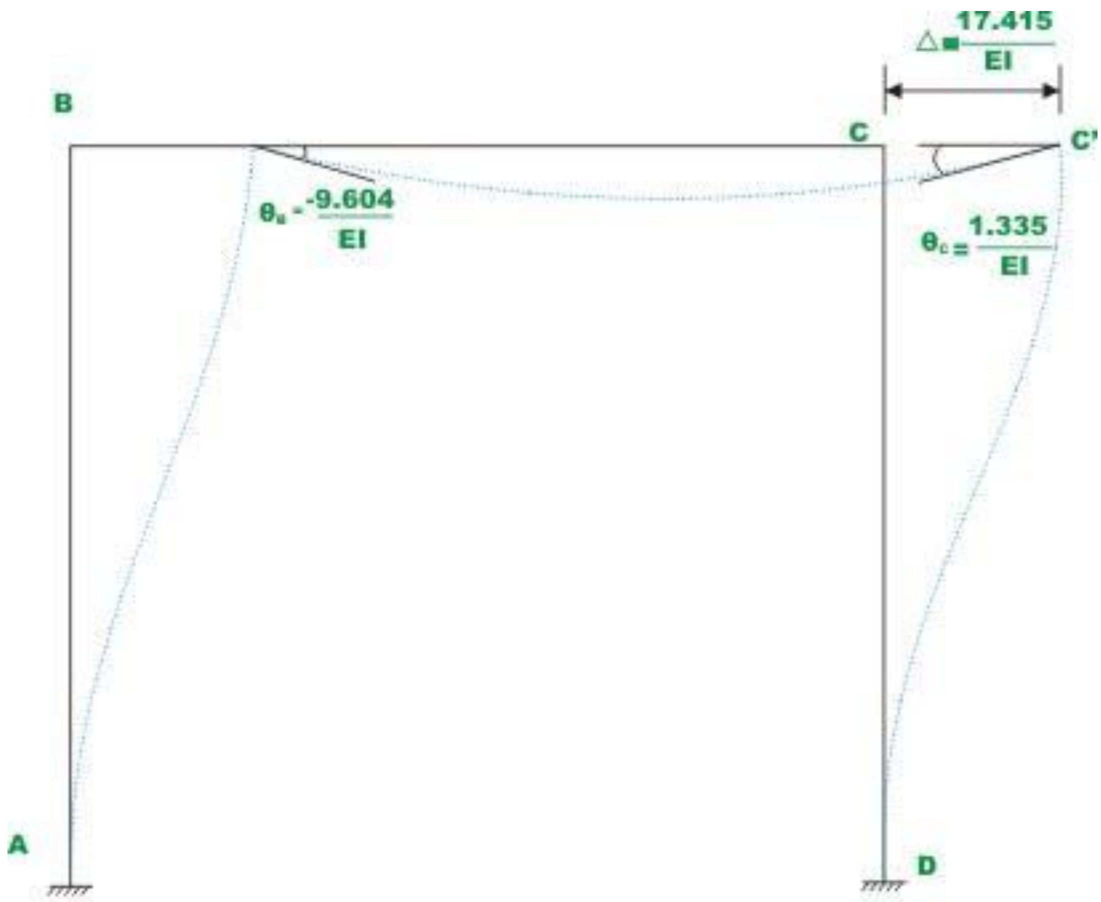


Fig.21.4e Elastic curve

Example 2

Analyse the rigid frame shown in Fig. 21.5a by moment-distribution method. The moment of inertia of all the members is shown in the figure. Neglect axial deformations.

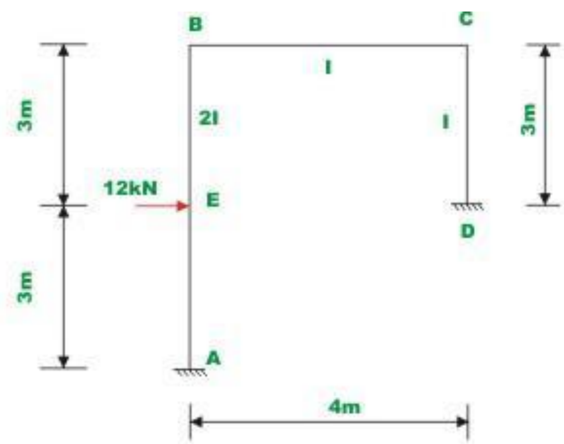


Fig. 21.5a Example 21.2

Solution:

In this frame joint rotations B and C and translation of joint B and C need to be evaluated.

a) Calculate stiffness and distribution factors.

$$K_{BA} = 0.333EI\,; \qquad K_{BC} = 0.25EI$$

$$K_{CB} = 0.25EI\,; \qquad K_{CD} = 0.333EI$$

At joint B :

$$\sum K = 0.583EI$$

$$DF_{BA} = 0.571\,; \qquad DF_{BC} = 0.429$$

At joint C :

$$\sum K = 0.583EI$$

$$DF_{CB} = 0.429 \, ; \qquad DF_{CD} = 0.571$$

b) Calculate fixed end moments due to applied loading.

$$M_{AB}^{F} = \frac{12 \times 3 \times 3^2}{6^2} = 9.0 \text{ kN.m} \, ; \; M_{BA}^{F} = -9.0 \text{ kN.m}$$

$$M_{BC}^{F} = 0 \text{ kN.m} \, ; \qquad M_{CB}^{F} = 0 \text{ kN.m}$$

$$M_{CD}^{F} = 0 \text{ kN.m} \, ; \qquad M_{DC}^{F} = 0 \text{ kN.m}$$

c) Prevent sidesway by providing artificial support at C . Carry out moment-distribution (*i.e.* Case A in Fig. 21.5b). The moment-distribution for this case is shown in Fig. 21.5c.

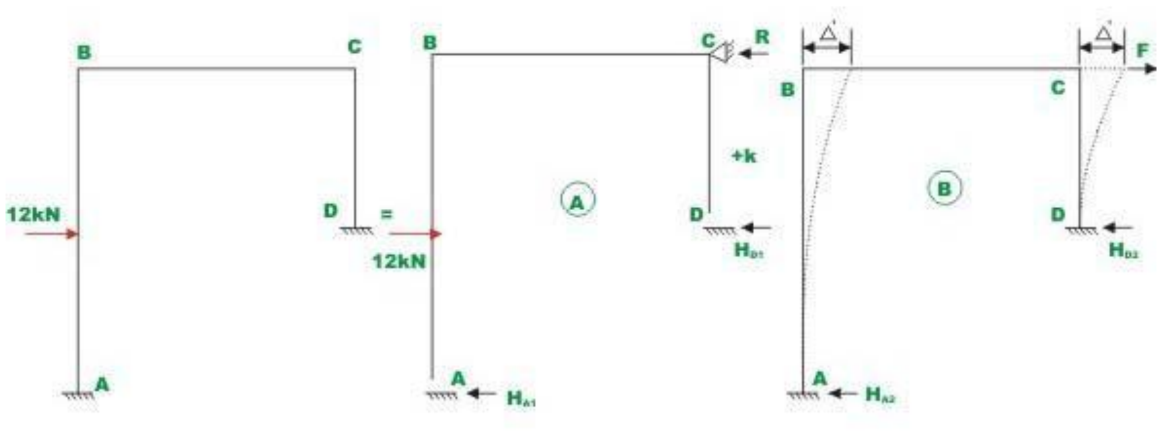


Fig. 21.5 b Frame with sidesway

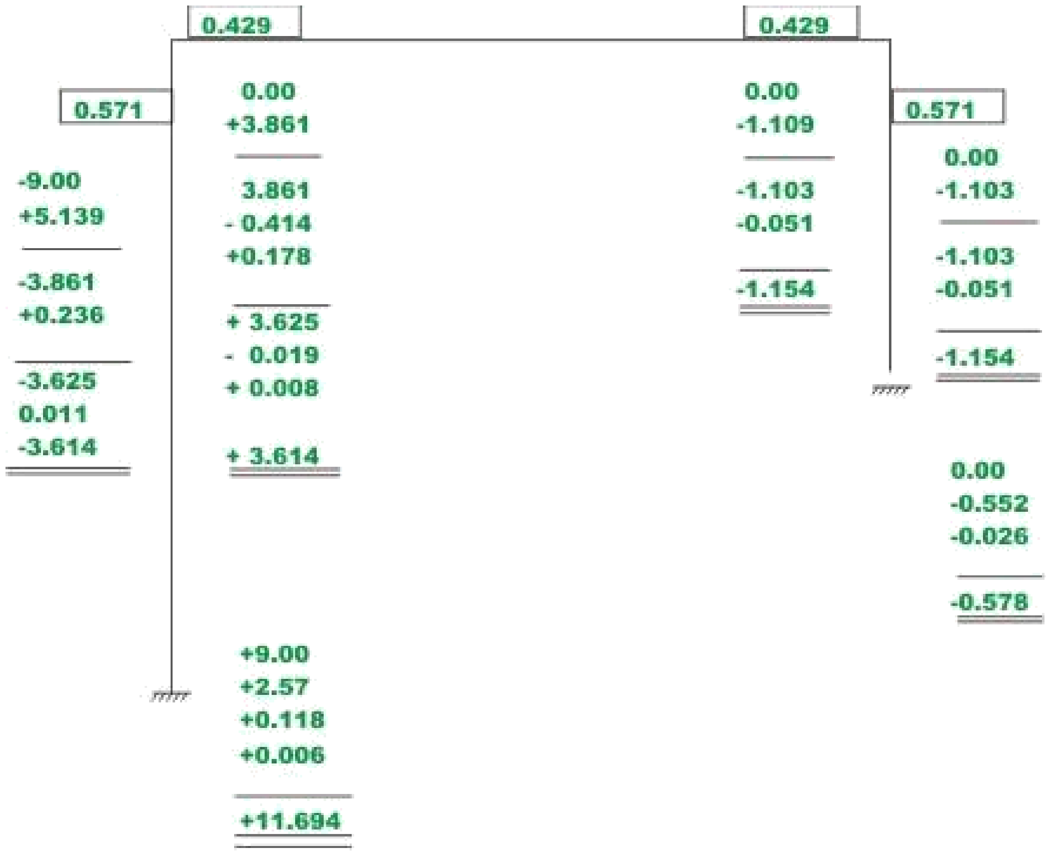


Fig. 21.5c Moment distribution with sidesway prevented

Now calculate horizontal reaction at A and D from equations of statics.

$$H_{A1} = \frac{11.694 - 3.614}{6} + 6 = 7.347 \quad \text{kN} \, (\leftarrow)$$

$$H_{D1} = \frac{-1.154 - 0.578}{3} = -0.577 \quad \text{kN} \, (\rightarrow)$$

$$R = 12 - (7.347 - 0.577) = -5.23 \quad \text{kN} \, (\rightarrow)$$

d) Moment-distribution for arbitrary sidesway ' (case B, Fig. 21.5c)

Calculate fixed end moments for the arbitrary sidesway of $' = \frac{150}{EI}$.

$$M_{AB}^{F} = -\frac{6E(2I)}{L}\psi = \frac{12EI}{6} \times \left(-\frac{150}{6EI}\right) = +50 \quad \text{kN.m} \;; \qquad M_{BA}^{F} = +50 \quad \text{kN.m} \;;$$

$$M_{CD}^F = -\frac{6E(I)}{L}\psi = -\frac{6EI}{3} \times \left(-\frac{150}{3EI}\right) = +100 \text{ kN.m} \; ; \qquad M_{DC}^F = +100 \text{ kN.m} \; ;$$

The moment-distribution for this case is shown in Fig. 21.5d. Using equations of static equilibrium, calculate reactions H_{A2} and H_{D2} .

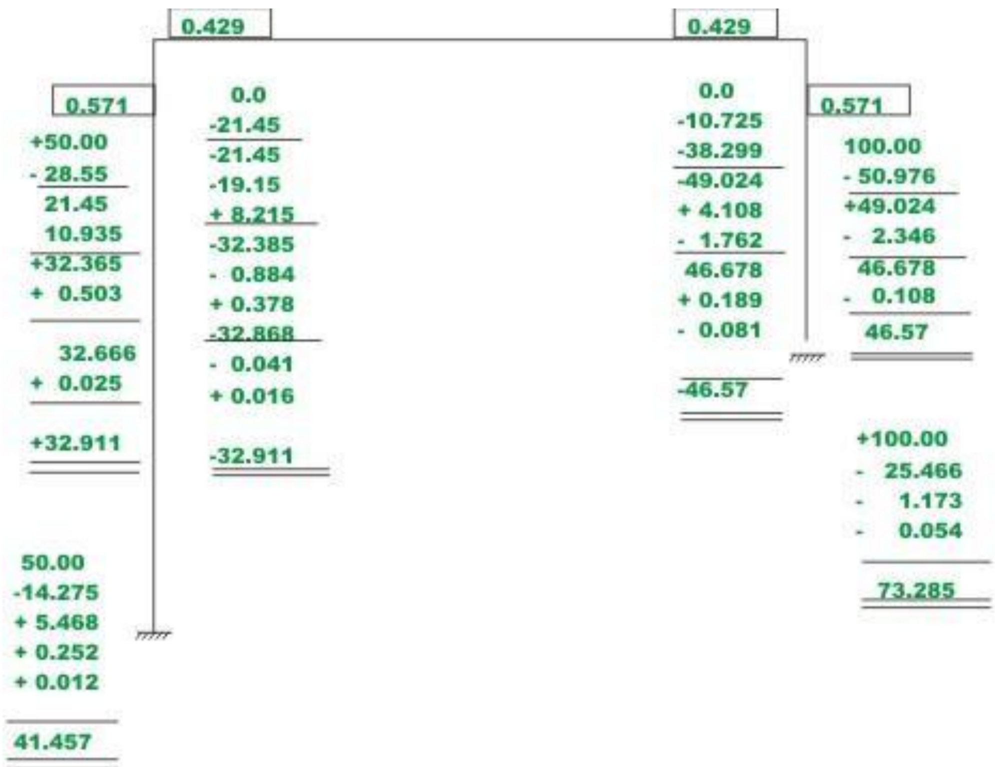


Fig. 21.5d Moment Distribution for arbitrary known sidesway

$$H_{A2} = \frac{32.911 + 41.457}{6} = 12.395 \; kN \; (\leftarrow)$$

$$H_{D2} = \frac{46.57 + 73.285}{3} = 39.952 \; kN \; (\leftarrow)$$

$$F = -(12.395 + 39.952) = -52.347 \; kN \; (\rightarrow)$$

e) Final results

Now, the shear condition for the frame is (vide Fig. 21.5b)

$$(H_{A1} + H_{D1}) + k(H_{A2} + H_{D2}) = 12$$

$$(7.344 - 0.577) + k(12.395 + 39.952)$$

$$= 12 \quad k = 0.129$$

Now the actual end moments in the frame are,

$$M_{AB} = M'_{AB} + k M''_{AB}$$

$$M_{AB} = 11.694 + 0.129(+41.457) = +17.039 \text{ kN.m}$$

$$M_{BA} = -3.614 + 0.129(+32.911) = 0.629 \text{ kN.m}$$

$$M_{BC} = 3.614 + 0.129(-32.911) = -0.629 \text{ kN.m}$$

$$M_{CB} = -1.154 + 0.129(-46.457) = -4.853 \text{ kN.m}$$

$$M_{CD} = -1.154 + 0.129(+46.457) = +4.853 \text{ kN.m}$$

$$M_{DC} = -0.578 + 0.129(+73.285) = +8.876 \text{ kN.m}$$

The actual sway

$$= k\,' = 0.129 \times \frac{150}{EI}$$

$$= \frac{19.35}{EI}$$

The joint rotations can be calculated using slope-deflection equations.

$$M_{AB} - M_{AB}^{F} = + \frac{2E(2I)}{L}\left[2\theta_A + \theta_B - 3\psi\right]$$

or

$$\left[2\theta_A + \theta_B\right] = \frac{L}{4EI}\left[M_{AB} - M_{AB}^{F} + \frac{12EI\psi}{L}\right] = \frac{L}{4EI}\left[M_{AB} - M_{AB}^{F} - \frac{12EI\psi}{L}\right]$$

$$\left[2\theta_B + \theta_A\right] = \frac{L}{4EI}\left[M_{BA} - M_{BA}^{F} + \frac{12EI\psi}{L}\right] = \frac{L}{4EI}\left[M_{BA} - M_{BA}^{F} - \frac{12EI\psi}{L}\right]$$

$$M_{AB} = +17.039 \text{ kN.m}$$

$$M_{BA} = 0.629 \quad kN.m$$

$$\left(M_{AB}^{F}\right) = 9 + 0.129(50) = 15.45 \quad kN.m$$

$$\left(M_{BA}^{F}\right) = -9 + 0.129(50) = -2.55 \quad kN.m$$

$$\theta_A = \frac{\text{change in near end} + -\frac{1}{2}\text{change in far end}}{3EI/L}$$

$$= \frac{(17.039 - 15.45) + -\frac{1}{2}(0.629 + 2.55)}{3EI/6} = 0.0$$

$$\theta_B = \frac{4.769}{EI}$$

Example 3

Analyse the rigid frame shown in Fig. 21.6a. The moment of inertia of all the members are shown in the figure.

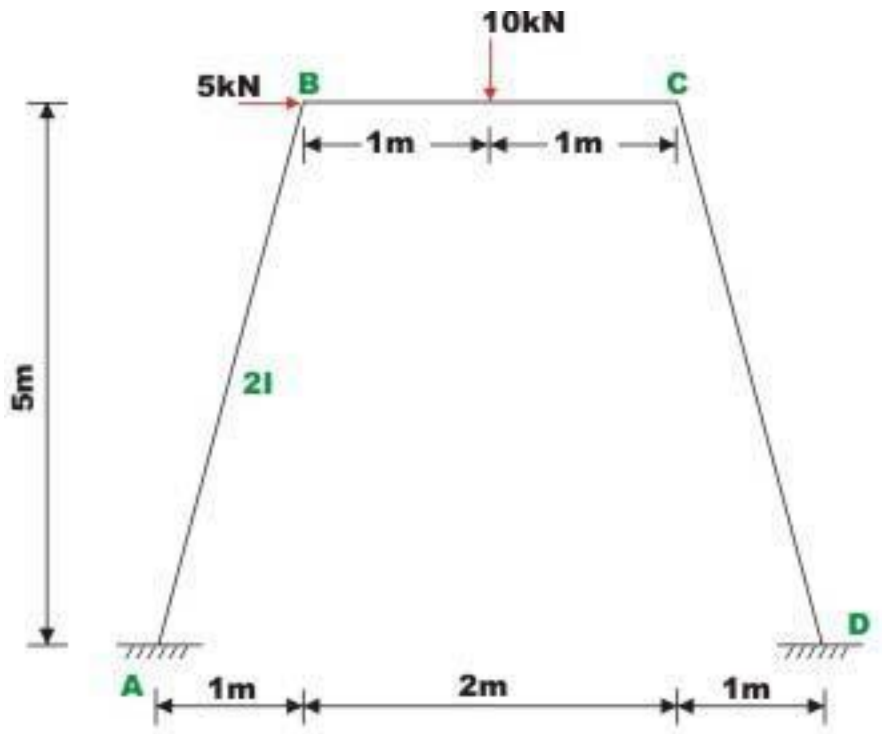


Fig.21.6a Example 21.3

Solution:

a) Calculate stiffness and distribution factors

$$K_{BA} = \frac{2EI}{5.1} = 0.392EI \; ; \qquad K_{BC} = 0.50EI$$

$$K_{CB} = 0.50EI \; ; \qquad K_{CD} = 0.392EI$$

At joint B :

$$\sum K = 0.892EI$$

$$DF_{BA} = 0.439 \; ; \qquad DF_{BC} = 0.561$$

At joint C :

$$\sum K = 0.892EI$$

$$DF_{CB} = 0.561 \; ; \qquad DF_{CD} = 0.439 \qquad (1)$$

b) Calculate fixed end moments due to applied loading.

$$M_{AB}^{F} = M_{BA}^{F} = M_{CD}^{F} = M_{DC}^{F} = 0 \quad \text{kN.m}$$

$$M_{BC}^{F} = 2.50 \quad \text{kN.m}$$

$$M_{CB}^{F} = -2.50 \quad \text{kN.m} \qquad (2)$$

c) Prevent sidesway by providing artificial support at C . Carry out moment-distribution for this case as shown in Fig. 21.6b.

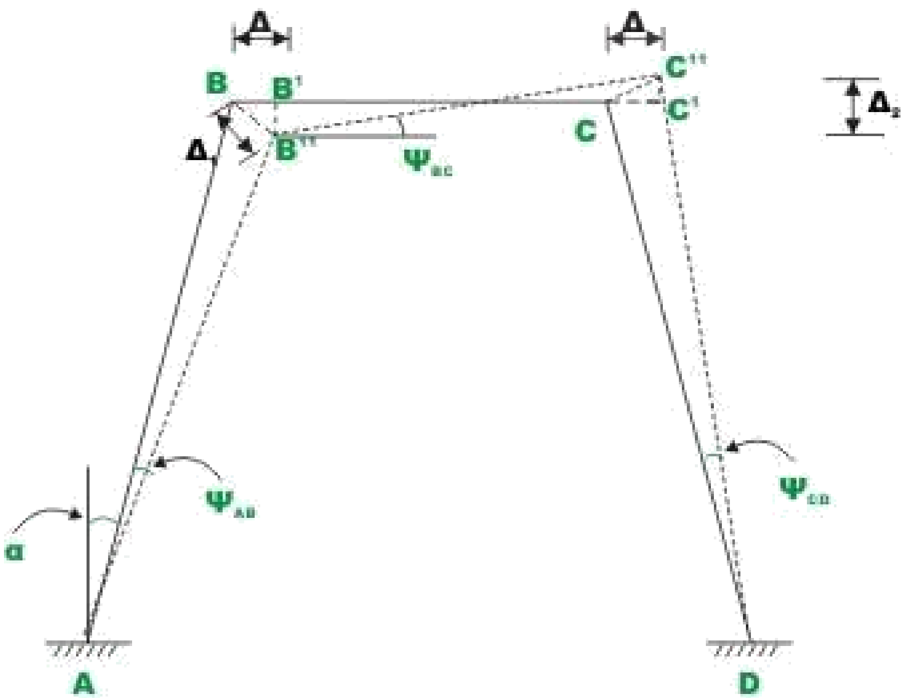


Fig.21.6b Rotation of Columns and beams

Now calculate reactions from free body diagram shown in Fig. 21.5d.

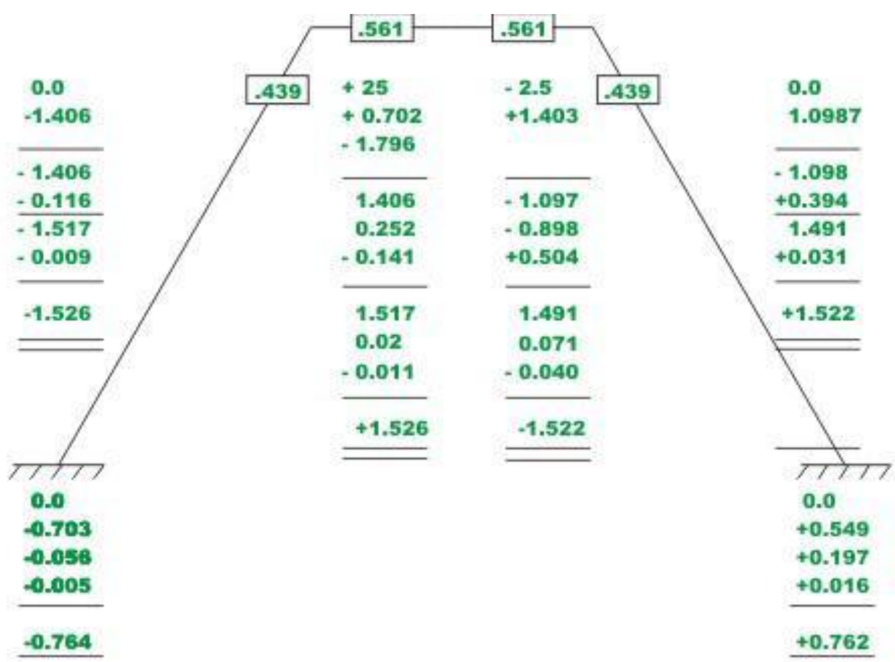


Fig. 21.6 © Moment distribution for applied loading

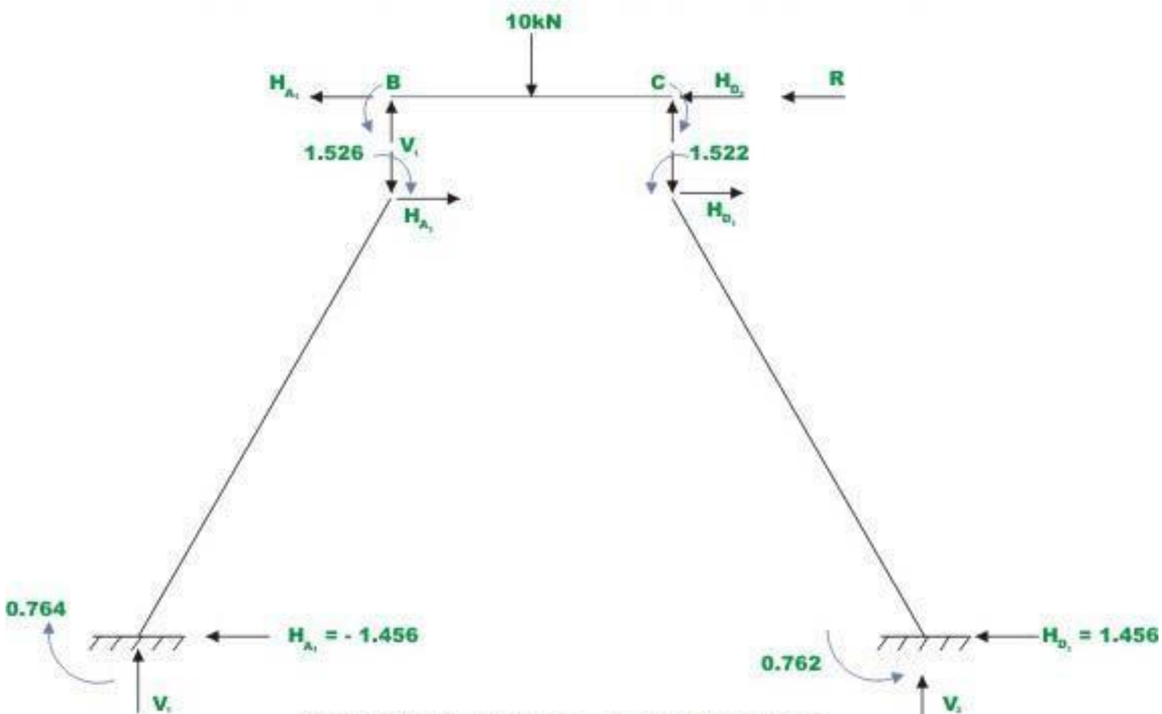


Fig. 21.6 (d) Free - body diagram

Column *AB*

$$\sum M_A = 0 \Rightarrow 5H_{A1} + 1.526 + 0.764 + V_1 = 0$$

$$5H_{A1} + V_1 = -2.29 \tag{3}$$

Column *CD*

$$\sum M_D = 0 \Rightarrow 5H_{D1} - 1.522 - 0.762 - V_2 = 0$$

$$5H_{D1} - V_2 = 2.284 \tag{4}$$

Beam *BC*

$$\sum M_C = 0 \Rightarrow 2V_1 + 1.522 - 1.526 - 10 \times 1 = 0$$

$$V_1 = 5.002 \text{ kN} (\uparrow)$$

$$V_2 = 4.998 \text{ kN} (\uparrow) \tag{5}$$

Thus from (3) $H_{A1} = -1.458 \text{ kN} (\rightarrow)$

from (4) $H_{D1} = 1.456 \text{ kN} (\leftarrow)$ (6)

$$\sum F_X = 0 \qquad H_{A1} + H_{D1} + R - 5 = 0 \qquad (7)$$

$$R = +5.002 \text{ kN } (\leftarrow)$$

d) Moment-distribution for arbitrary sidesway ' .

Calculate fixed end beam moments for arbitrary sidesway of

$$' = \frac{12.75}{EI}$$

The member rotations for this arbitrary sidesway is shown in Fig. 21.6e.

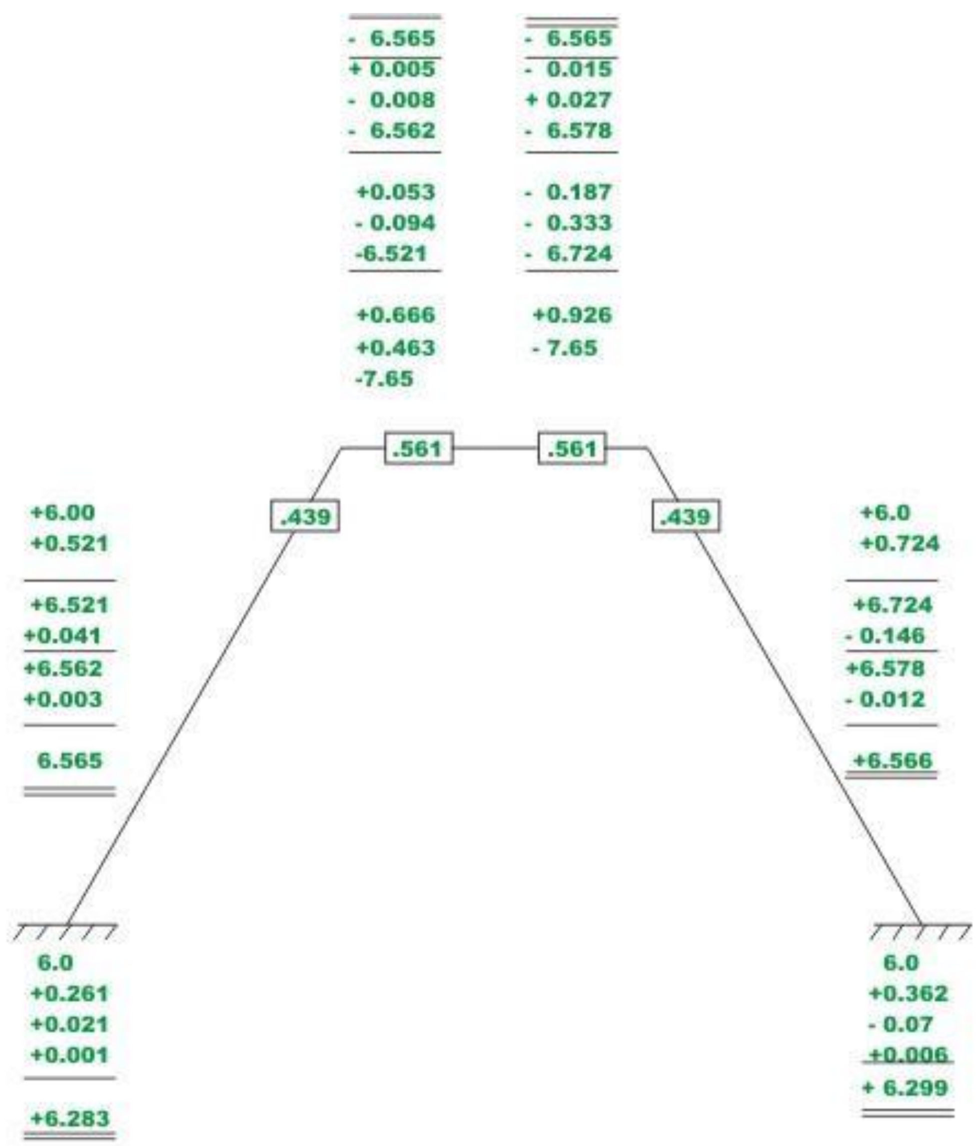


Fig. 21.6 (e) Moment distribution of arbitrary known sidesway

$$\psi_{AB} = \frac{BB''}{L_{AB}} = -\frac{\Delta_1}{L_{AB}} \; ; \qquad \Delta_1 = \frac{\Delta'}{\cos\alpha} = \frac{5.1\,\Delta'}{5}$$

$$\Delta_2 = \frac{2\,\Delta'}{5} = 0.4\,\Delta'$$

$$\psi_{AB} = -\frac{\Delta'}{5} \,(\textit{clockwise}) \; ; \psi_{CD} = -\frac{\Delta'}{5} \,(\textit{clockwise})$$

$$\psi_{BC} = \frac{\Delta_2}{2} = \frac{2\,\Delta' \tan\alpha}{2} = \frac{\Delta'}{5} \,(\textit{counterclockwise})$$

$$M_{AB}^{F} = -\frac{6\,EI_{AB}}{L_{AB}}\psi_{AB} = -\frac{6\,E\,(2\,I\,)}{5.1} - \frac{12.75}{5EI} = +6.0 \text{ kN.m}$$

$$M_{BA}^{F} = +6.0 \text{ kN.m}$$

$$M_{BC}^{F} = -\frac{6\,EI_{BC}}{L_{BC}}\psi_{BC} = -\frac{6E\,(\,I\,)}{2}\,\frac{12.75}{5EI} = -7.65 \text{ kN.m}$$

$$M_{CB}^{F} = -7.65 \text{ kN.m}$$

$$M_{CD}^{F} = -\frac{6\,EI_{CD}}{L_{CD}}\psi_{CD} = -\frac{6\,E\,(2I\,)}{5.1} - \frac{12.75}{5EI} = +6.0 \text{ kN.m}$$

$$M_{DC}^{F} = +6.0 \text{ kN.m}$$

The moment-distribution for the arbitrary sway is shown in Fig. 21.6f. Now reactions can be calculated from statics.

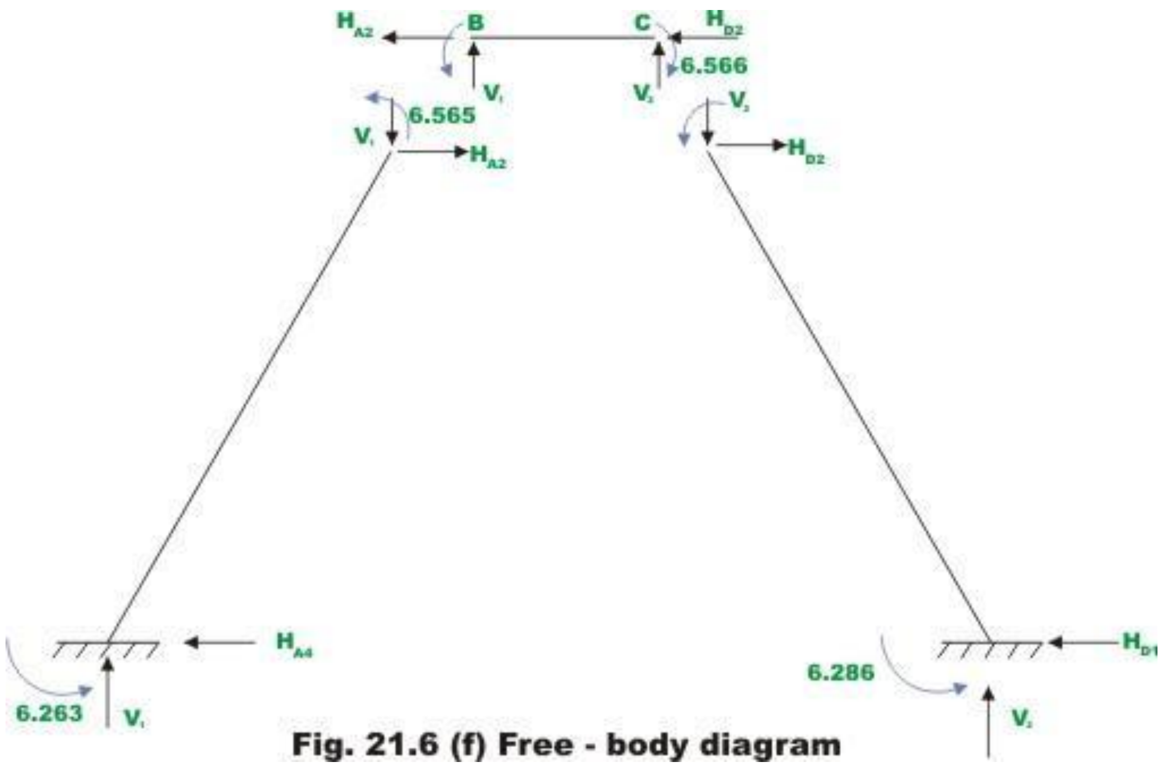


Fig. 21.6 (f) Free - body diagram

Column *AB*

$$\sum M_A = 0 \Rightarrow 5H_{A2} - 6.283 - 6.567 + V_1 = 0$$

$$5H_{A1} + V_1 = 12.85 \qquad (3)$$

Column *CD*

$$\sum M_D = 0 \Rightarrow 5H_{D2} - 6.567 - 6.283 - V_2 = 0$$

$$5H_{D1} - V_2 = 12.85 \qquad (4)$$

Beam *BC*

$$\sum M_C = 0 \Rightarrow 2V_1 + 6.567 + 6.567 = 0$$

$$V_1 = -6.567 \ kN (\downarrow); V_2 = +6.567 \ \text{kN} (\uparrow) \qquad (5)$$

Thus from 3 $H_{A2} = +3.883 \ \text{kN} (\leftarrow)$

from 4 $H_{D2} = 3.883 \ \text{kN} (\leftarrow)$ (6)

$$F = 7.766 \quad \text{kN} \ (\leftarrow) \tag{7}$$

e) **Final results**

$$k\,F = R$$

$$k = \frac{5.002}{7.766} = 0.644$$

Now the actual end moments in the frame are,

$$M_{AB} = M'_{AB} + k\,M''_{AB}$$

$$M_{AB} = -0.764 + 0.644(+6.283) = +3.282 \quad \text{kN.m}$$

$$M_{BA} = -1.526 + 0.644(+6.567) = 2.703 \quad \text{kN.m}$$

$$M_{BC} = 1.526 + 0.644(-6.567) = -2.703 \quad \text{kN.m}$$

$$M_{CB} = -1.522 + 0.644(-6.567) = --5.751 \quad \text{kN.m}$$

$$M_{CD} = 1.522 + 0.644(6.567) = 5.751 \quad \text{kN.m}$$

$$M_{DC} = 0.762 + 0.644(6.283) = 4.808 \quad \text{kN.m}$$

UNIT- V

KANIS METHOD

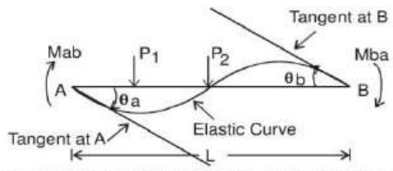
This method may be considered as a further simplification of moment distribution method wherein the problems involving sway were attempted in a tabular form thrice (for double story frames) and two shear co-efficients had to be determined which when inserted in end moments gave us the final end moments. All this effort can be cut short very considerably by using this method.

→ Frame analysis is carried out by solving the slope – deflection equations by successive approximations. Useful in case of side sway as well.

→ Operation is simple, as it is carried out in a specific direction. If some error is committed, it will be eliminated in subsequent cycles if the restraining moments and distribution factors have been determined correctly. Please note that the method does not give realistic results in cases of columns of unequal heights within a storey and for pin ended columns both of these cases are in fact extremely rare even in actual practice. Even codes suggest that RC columns framing into footings or members above may be considered more or less as fixed for analysis and design purposes.

Case 1. No side sway and therefore no translation of joints derivation.

Consider a typical member AB loaded as shown below:



A GENERAL BEAM ELEMENT UNDER END MOMENTS AND LOADS

General Slope deflection equations are.

$$Mab = MFab + \frac{2EI}{L}(-2\theta a - \theta b) \quad \rightarrow (1)$$

$$Mba = MFba + \frac{2EI}{L}(-\theta a - 2\theta b) \quad \rightarrow (2)$$

equation (1) can be re-written as

$$Mab = MFab + 2\,M'ab + M'ba \quad \rightarrow (3)$$ where MFab = fixed end moment at A due to applied loads.

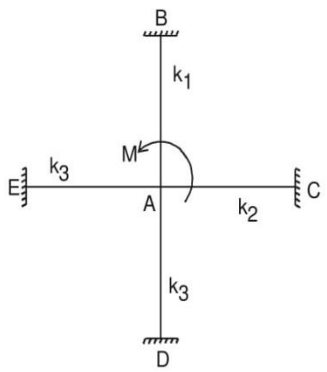
and $M'ab$ = rotation contribution of near end A of member AB $= -\frac{EI}{L}(2\theta a)$

$$= -\frac{2EI\,\theta a}{L} = -2E\,k_1\,\theta a \quad \rightarrow (4) \text{ where } \left[k_1 = \frac{I_1}{L_1}\right]$$

$M'ba$ = rotation contribution of for end B of member AB.

So $$M'ba = -\frac{2\,EI\,\theta b}{L} = -2Ek_1\,\theta b \quad \rightarrow (5)$$

Now consider a generalized joint A in a frame where members AB, AC, AD.........meet. It carries a moment M.



For equilibrium of joint A, $\sum Ma = 0$

or Mab + Mac + Mad + Mae..................= 0 Putting these end moments in form of eqn. (3)

or $\sum MF\,(ab, ac, ad) + 2\sum M'\,(ab, ac, ad) + \sum M'\,(ba, ca, da) = 0$

Let $\sum MF\,(ab, ac, ad) = MFa$ (net FEM at A)

So $MFa + 2\sum M'\,(ab, ac, ad) + \sum M'\,(ba, ca, da) = 0 \quad \rightarrow (6)$

From (6), $\sum M'\,(ab, ac, ad) = -\frac{1}{2}[(MFa + \sum M'\,(ba, ca, da)] \quad \rightarrow (7)$

From (4), $\sum M'\,(ab, ac, ad) = -2Ek_1\,\theta a - 2\,Ek_2\,\theta a - 2\,Ek_3\,\theta a + \ldots\ldots\ldots\ldots\ldots$

$= -2\,E\theta a\,(k_1 + k_2 + k_3)$

$= -2\,E\theta a\,(\sum k)$, (sum of the member stiffnesses framing in at joint A)

$$\text{or} \quad \theta a = -\frac{\sum M'\,(ab, ac, ad)}{2E\,(\sum k)} \quad \rightarrow (8)$$

From (4), $M'ab = -2\,Ek_1\,\theta a$. Put θa from (8), we have

$$M'ab = -2E\,k_1\left[-\frac{\sum M'\,(ab, ac, ad)}{2E\,(\sum k)}\right] = \frac{k_1}{\sum k}[\,\sum M'\,(ab, ac, ad)]$$

From (7), Put $\sum M'\,(ab, ac, ad)$

So $M'ab = \frac{k_1}{\sum k}\left[-\frac{1}{2}(MFa + \sum M'\,(ba, ca, da))\right]$

or $M'ab = -\frac{1}{2}\frac{k_1}{\sum k}$ [MFa $+ \sum M'$ (ba, ca, da)]

on similar lines $M'ac = -\frac{1}{2}\frac{k_2}{\sum k}$ [MFa $+ \sum M'$ (ba, ca, da)]

and $M'ad = -\frac{1}{2}\frac{k_3}{\sum k}$ [MFa $+ \sum M'$ (ba, ca, da)]

↙ rotation contribution of near end of member ad.

↙ sum of the rotations contributions of far ends of members meeting at A.

Sum of rotation factors at near end of members ab, ac, ad is

$$-\frac{1}{2}\frac{k_1}{\sum k} - \frac{1}{2}\frac{k_2}{\sum k} - \frac{1}{2}\frac{k_3}{\sum k} = -\frac{1}{2}\left[\frac{k_1 + k_2 + k_3 + \ldots\ldots}{\sum k}\right]$$

$= -\frac{1}{2}$, [sum of rotation factors of different members meeting at a joint is equal to $-\frac{1}{2}$]

Therefore, if net fixed end moment at any joint along with sum of the far end contribution of members meeting at that joint are known then near end moment contribution can be determined. If far end contributions are approximate, near end contributions will also be approximate. When Far end contributions are not known (as in the first cycle), they can be assumed to be zero.

6.1. RULES FOR CALCULATING ROTATION CONTRIBUTIONS :— Case-1: Without sides way.

Definition: **"Restrained moment at a joint is the algebraic sum of FE.M's of different members meeting at that joint."**

1. Sum of the restrained moment of a joint and all rotation contributions of the far ends of members meeting at that joint is multiplied by respective rotation factors to get the required near end rotation contribution. For the first cycle when far end contributions are not known, they may be taken as zero (Ist approximation).

2. By repeated application of this calculation procedure and proceeding from joint to joint in an arbitrary sequence **but in a specific direction**, all rotation contributions are known. The process is usually stopped when end moment values converge. This normally happens after three or four cycles. But values after 2nd cycle may also be acceptable for academic.

6.2. Case 2:— **With side sway (joint translations)**

In this case in addition to rotation contribution, linear displacement contributions (Sway contributions) of columns of a particular storey are calculated after every cycle as follows:

6.2.1. For the first cycle.

(A) → **Linear Displacement Contribution** (LDC) of a column = Linear displacement factor (LDF) of a particular column of a story multiplied by [storey moment + contributions at the ends of columns of that story]

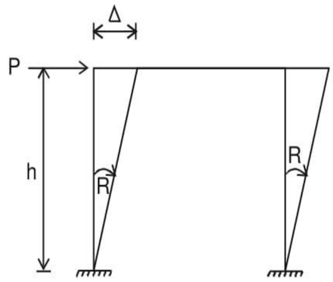
Linear displacement factor (LDF) for columns of a storey = $-\frac{3}{2}$

Linear displacement factor of a column = $-\frac{3}{2}\frac{k}{\Sigma k}$ Where k=stiffness of the column being considered and Σk is the sum of stiffness of all columns of that storey.

6.2.2. (B) → **Storey moment** = Storey shear x $\frac{1}{3}$ of storey height.

6.2.3. (C) → **Storey shear** : It may be considered as reaction of column at horizontal beam / slab levels due to lateral loads by considering the columns of each sotrey as simply supported beams in vertical direction. **"If applied load gives + R value (according to sign conversion of slope deflection method), storey shear is +ve or vice versa."**

Consider a general sway case.



6.3. SIGN CONVENSION ON MOMENTS:– Counter-clockwise moments are positive and clockwise rotations are positive.

For first cycle with side sway.

(D) Near end contribution of various members meeting at that joint. = respective rotation contribution factor × [Restrained moment + far end contributions]

Linear displacement contributions will be calculated after the end of each cycle for the columns only.

FOR 2ND AND SUBSEQUENT CYCLES.

(E) → Near end contributions of various members meeting at a joint. = Respective rotation contribution factor × [Restrained moment + far end contributions + linear displacement contribution of columns of different storeys meeting at that joint].

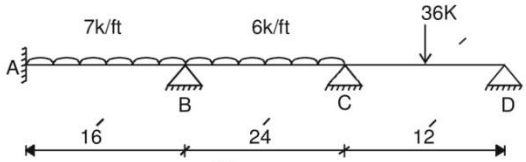
6.4. Rules for the Calculation of final end moments (sidesway cases)

(F) For beams, End moment = FEM + 2 near end contribution + Far end contributions.

(G) For columns, End moment. = FEM + 2 near end contribution + Far end contribution + linear displacement contribution of that column for the latest cycle.

6.5. APPLICATION OF ROTATION CONTRIBUTION METHOD (KANI'S METHOD) FOR THE ANALYSIS OF CONTINUOUS BEAMS

Example No.1: Analyze the following beam by rotation contribution method. EI is constant.



EI = constt.

Note. Analysis assumes continuous ends with some fixity. Therefore, in case of extreme hinged supports in exterior spans, modify (reduce) the stiffness by 3/4 = (0.75).for a hinged end.

Step No. 1. Relative Stiffness.

Span	I	L	$\frac{I}{L}$	K_{rel}	K modified.
AB	1	16	$\frac{1}{16} \times 48$	3	3
BC	1	24	$\frac{1}{24}$	2	2
CD	1	12	$\frac{1}{12}$	4 x (3/4)	3

(exterior or discontinuous hinged end)

Step No.2. Fixed end moments.

$$Mfab = +\frac{wL^2}{12} = +\frac{3 \times 16^2}{12} = +64 \text{ K-ft.}$$

$$Mfba = -64$$

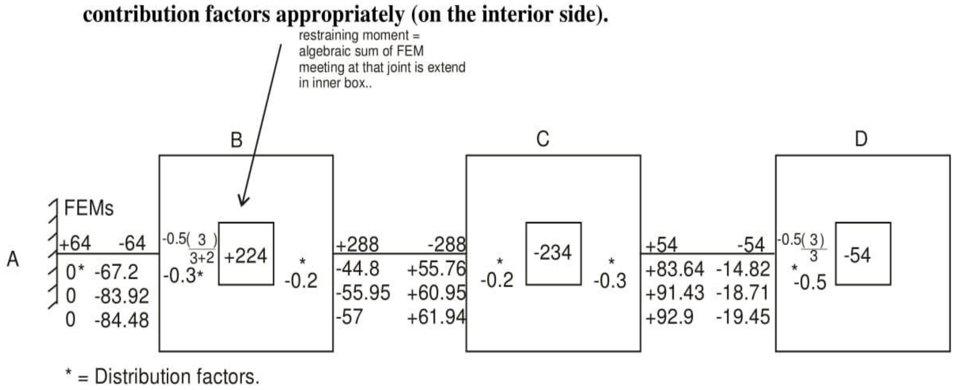
$$Mfbc = +\frac{6 \times 24^2}{12} = +288$$

$$Mfcb = -288$$

$$Mfcd = +\frac{Pa^2b}{L^2} = \frac{+36 \times 6^2 \times 6}{12^2} = +54$$

$$Mfdc = -54$$

Step No.3. Draw Boxes, enter the values of FEMs near respective ends of exterior boxes and rotation contribution factors appropriately (on the interior side).



FIRST CYCLE

A ↓ C(Far end contribution) ↓   B ↓ D(Far end contributions) ↓

Joint B: − 0.3 (+224 + 0 + 0) = − 67.2 (Span BA)   Joint C: − 0.2(− 234 − 44.8 + 0) = +55.76 (Span CB)

and   − 0.2 (224 + 0 + 0) = − 44.8 (Span BC)   and   − 0.3(− 234 − 44.8 + 0) = +83.64 (Span CD)

Joint D: − 0.5(− 54 +83.64) = − 14.82 (Span DC)

2nd cycle:

A ↓ C (Far end contributions) ↓   B ↓ D (far end contributions) ↓

Joint B. − 0.3 (+ 224+0 +55.76) = − 83.92   Joint C: − 0.2 (− 234 − 55.95 − 14.82) = 60.95

− 0.2 (+224+0 +55.76) = − 55.85   − 0.3 (− 234 − 55.95 − 14.82) = 91.43

Joint D. − 0.5 (− 54 + 91.43) = − 18.715

3rd cycle: Singular to second cycle procedure. We stop usually after 3 cycles and the answers can be further refined by having another couple of cycles. (Preferably go up to six cycles till difference in moment value is 0.1 or less). The last line gives near and far end contribution.

Step No. 4. FINAL END MOMENTS

For beams. End moment = FEM + 2near end cont. + Far end contribution.

Mab = + 64 + 2 x 0 − 84.48 = − 20.48 k − ft.

Mba = − 64 − 2 x 84.48 + 0 = − 232.96 k − ft.

Mbc = + 288− 2 x 57 + 61.94 = +235.9 k − ft.

Mcb = − 288 + 2 x 61.94 − 57 = − 221.12

Mcd = + 54 + 2 x 92.9 − 19.45 = + 220.35

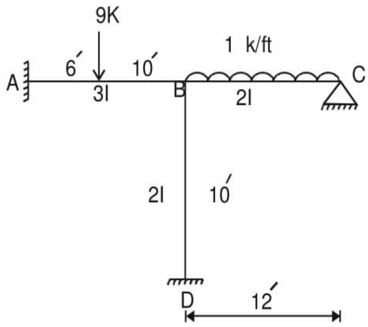
Mdc = − 54 − 2 x 19.45 + 92.9 = zero

The beam has been analyzed and we can draw shear force and bending moment diagrams as usual.

6.6. Rotation Contribution Method: Application to frames without side sway.

Example No 2:

Analyze the following frame by Kanis method (rotation Contribution Method)



Step No. 1 **Relative Stiffness.**

Span	I	L	$\frac{I}{L}$	K_{rel}	K modified.
AB	3	16	$\frac{3}{16}\times 240$	45	45
BC	2	12	$\frac{2}{12}\times 240$	$40\left(\frac{3}{4}\right)$	30 (Exterior hinged end)
BD	2	10	$\frac{2}{10}\times 240$	48	48
					$\Sigma 103$

Step No.2. **FEM's**

$$Mfab = \frac{9\times 6\times 10^2}{16^2} = +21.1 \text{ K-ft}$$

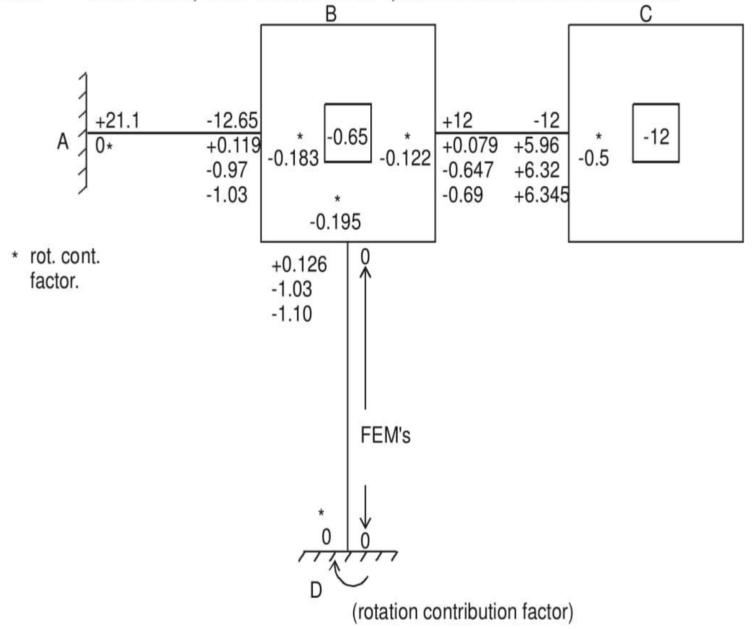
$$Mfba = \frac{9\times 10\times 6^2}{16^2} = -12.65$$

$$Mfbc = \frac{1\times 12^2}{12} = +12$$

$$Mfcb = -12$$

$Mfbd = Mfdb = 0$ (No load within span BD)

Step No. 3. **Draw Boxes, enter values of FEM's, rotation contribution factors etc.**



Apply all relevant rules in three cycles. Final end moments may now be calculated.

For beams. End moment = FEM + 2 x near end contribution. + Far end contribution

For Columns : End moment = FEM + 2 x near end contribution + Far end contribution + Linear displacement contribution of that column. To be taken in sway cases only.

Mab = 21.1 + 2x0 –1.03 = + 20.07 K–ft

Mba = –12.65 –2 x 1.03 + 0 = –14.71

Mbc = +12 –2 x 0.69 + 6.345 = 16.965

Mbd = 0 – 2x1.1 +0 = –2.2

Mcb = –12 + 2x 6.345 –0.69 = 0

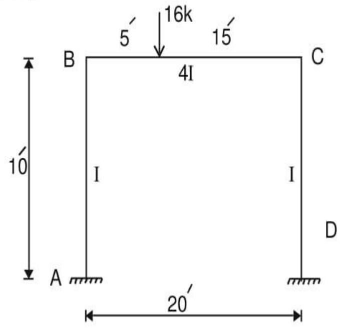
Mdb = 0 + 2x0–1.10 = –1.10

Equilibrium checks are satisfied. End moment values are OK. Now SFD and BMD can be drawn as usual.

Example No. 3: Analyse the following frame by rotation Contribution Method.

SOLUTION:-

It can be seen that sway case is there.



Step No. 1. **Relative Stiffness.**

Member.	I	L	$\frac{I}{L}$	K_{rel}
AB	1	10	$\frac{1}{10} \times 10$	1
BC	4	20	$\frac{4}{20} \times 10$	2
CD	1	10	$\frac{1}{10} \times 10$	1

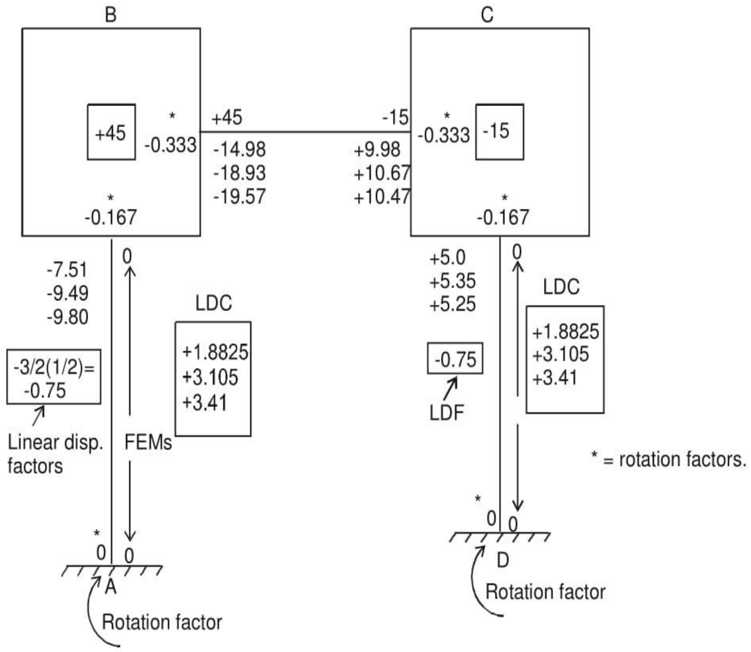
Step No. 2. **FEM's**

$$Mf_{BC} = \frac{+16 \times 5 \times 15^2}{20^2} = +45$$

$$Mf_{CB} = \frac{-16 \times 5^2 \times 15}{20^2} = -15$$

All other fixing moments are zero.

Step No.3 **Draw Boxes, enter FEM's and rotation Contribution factors etc. Apply three cycles.**



See explanation of calculations on next page.

Note: After applying the first cycle as usual, calculate linear displacement contribution for columns of all storeys. Repeat this calculation after every cycle.

Linear displacement contribution (LDC) of a column=Linear displacement factor [story moment + contribution of column ends of that storey)

Storey moment is zero because no horizontal load acts in column and there is no storey shear.

↓

After 1st cycle: Linear Disp. Cont = $-0.75\,[\,0 + 5.0 - 7.5 + 0 + 0] = +1.8825$

→ **For 2nd cycle onwards to calculate rotation contribution, apply following Rule**:–

Rotation contribution = rotation contribution factor [restrained moment + far end contributions + linear displacement contribution of columns. of different. storeys meeting at that joint.]

2nd cycle.

A C(Far ends)
↓ ↓

Joint B. $-0.167 [+45 + 0 + 9.98 + 1.8825] = -9.49$ (Span BA)

and -0.333 [——— do ————] = − 18.93 (Span BC)

Joint C. $-0.333 [-15 - 18.93 + 0 + 1.8825] = +10.67$ (Span CB)

and -0.167 [———— do ————] = + 5.35 (Span CD)

After 2nd cycle. Linear displacement contribution is equall to storey moment.

↓

$= -0.75 [0 - 9.49 + 0 + 5.35 + 0] = +3.105$

After 3rd cycle.

After 3rd cycle , linear displacement. contribution of columns is equall to storey moment.

↓

$= -0.75 [0 - 9.80 + 5.25 + 0 + 0] = 3.41$

Calculate end moments after 3rd cycle.

For beams: End moment = FEM + 2 near end contribution. + Far end contribution.

For columns. End moment = FEM + 2 near end contribution + Far end contribution. + linear displacement. contribution of that column.

Applying these rules

$M_{ab} = 0 + 0 - 9.80 + 3.41 = -6.3875$ k.ft.

$M_{ba} = +0 - 2 \times 9.80 + 0 + 3.41 = +16.19$

$M_{bc} = +45 - 2 \times 19.57 + 10.47 = +16.33$

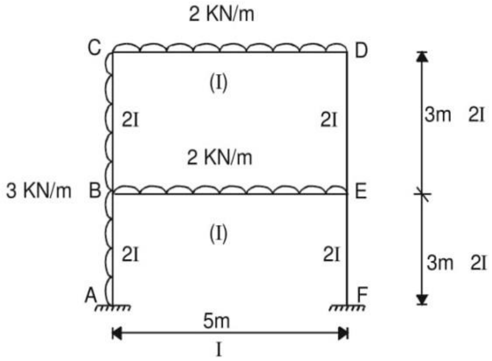
$M_{cb} = -15 + 2 \times 10.47 - 19.57 = 13.63$

$M_{cd} = 0 + 2 \times 5.25 + 0 + 3.41 = 13.91$

$M_{dc} = 0 + 2 \times 0 + 5.25 + 3.41 = 8.66$

By increasing number of cycles the accuracy is increased.

Example No 4 : Solve the following double story frame carrying gravity and lateral loads by rotation contribution method.



SOLUTION :–

If this is analyzed by slope-deflection or Moment distribution method, it becomes very lengthy and laborious. This becomes easier if solved by rotation contribution method.

Step 1: **F.E.Ms.**

$$Mfab = \frac{+3 \times 3^2}{12} = +2.25 \text{ KN–m}$$

Mfba = – 2.25 KN–m

Mfbc = + 2.25 KN–m

Mfcb = – 2.25 KN–m

$$Mfcd = \frac{2 \times 5^2}{12} = +4.17 \text{ KN–m}$$

Mfdc = – 4.17 KN–m

Mfbe = + 4.17 KN–m

Mfeb = – 4.17 KN–m.

Mfde = Mfed = 0

Mfef = Mffe = 0

Step 2: **RELATIVE STIFFNESS :–**

Span	I	L	$\frac{I}{L}$	K
AB	2	3	$\frac{2}{3} \times 15$	10
BC	2	3	$\frac{2}{3} \times 15$	10

BE	1	5	$\frac{1}{5} \times 15$	3
CD	1	5	$\frac{1}{5} \times 15$	3
DF	2	3	$\frac{2}{3} \times 15$	10
EF	2	3	$\frac{2}{3} \times 15$	10

LINEAR DISPLACEMENT FACTOR = L.D.F. of a column of a particular storey.

$$\text{L.D.F.} = -\frac{3}{2}\frac{K}{\sum K}$$

Where K is the stiffness of that column & $\sum K$ is the stiffness of columns of that storey. Assuming columns of equal sizes in a story. (EI same)

$$\text{L.D.F}_1 = -\frac{3}{2} \times \frac{10}{(10+10)} = -0.75 \qquad \text{(For story No. 1)}$$

$$\text{L.D.F}_2 = -\frac{3}{2} \times \frac{10}{(10+10)} = -0.75 \qquad \text{(For story No. 2)}$$

Storey Shear :–

This is, in fact, reaction at the slab or beam level due to horizontal forces. If storey shear causes a (–ve) value of R, it will be (–ve) & vice versa.

For determining storey shear the columns can be treated as simply supported vertical beams.

(1) Storey shear = – 9 KN (For lower or ground story. At the slab level of ground story)
(2) Storey shear = – 4.5 (For upper story). At the slab level of upper story root)

Storey Moment (S.M) :–

S.M. = Storey shear + h/3 where h is the height of that storey.

$$SM_1 = -9 \times \frac{3}{3} = -9 \qquad \text{(lower story)}$$

$$S.M_2 = -4.5 \times \frac{3}{3} = -4.5 \qquad \text{(Upper story)}$$

Rotation Factors

The sum of rotation factors at a joint is – ½. The rotation factors are obtained by dividing the value – ½ between different members meeting at a joint in proportion to their K values.

$$\mu ab = -\frac{1}{2}\frac{k_1}{\sum k}$$

$$\mu ac = -\frac{1}{2}\frac{k_2}{\sum k} \text{ etc.}$$

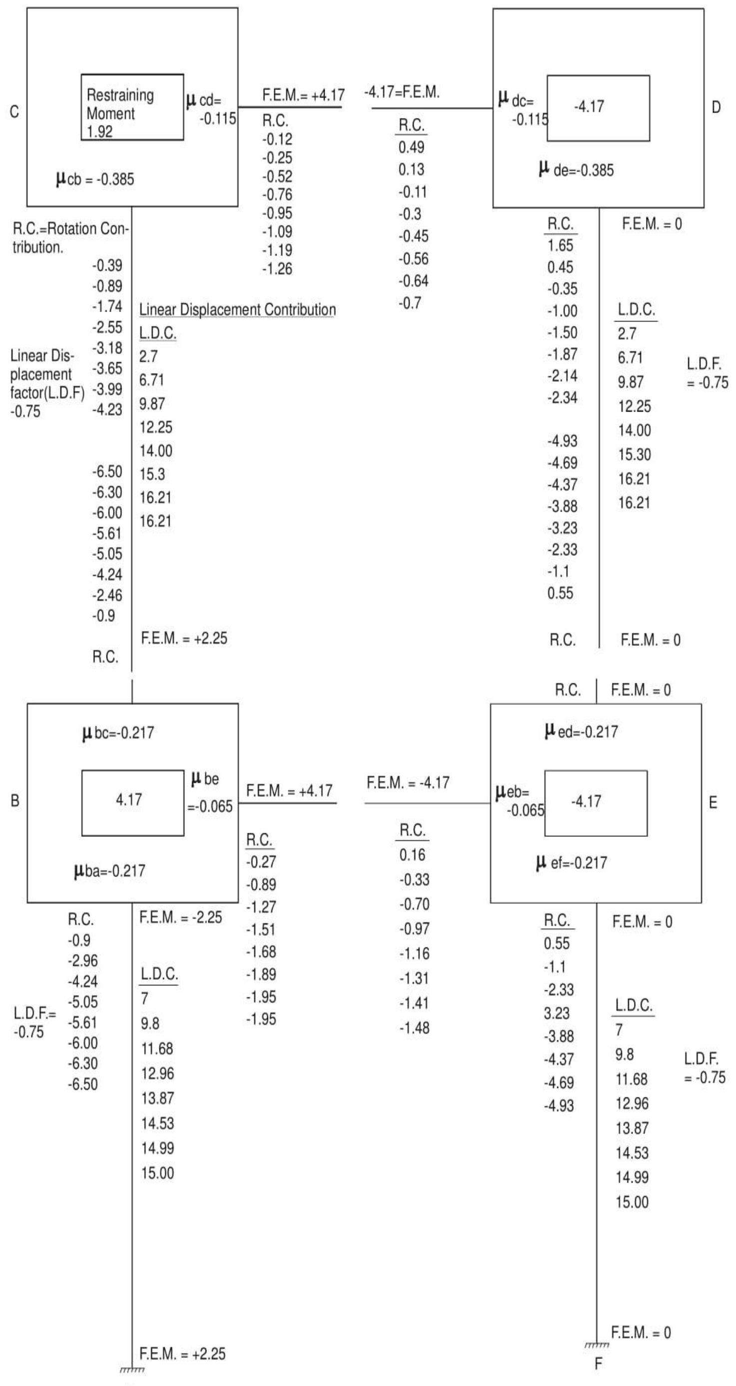
Rotation Contributions:–

The rule for calculating rotation contribution is as follows.

Sum the restrained moments of a point and all rotation contribution of the far ends of the members meeting at a joint. Multiply this sum by respective rotation factors to get the required rotation contribution. For the first cycle far end contribution can be taken as zero.

Span	K	Rotation factor.
AB	10	0 (Being fixed end)
BC	10	$-\frac{1}{2}\left(\frac{10}{23}\right) = -0.217$
BE	3	$-0.5\left(\frac{3}{23}\right) = -0.065$
BA	10	$-0.5\left(\frac{10}{23}\right) = -0.217$
CB	10	– 0.385
CD	3	– 0.115
DC	3	– 0.115
DE	10	– 0.385
ED	10	– 0.217
EB	3	– 0.065
EF	10	– 0.217
FE	10	0 (Being fixed end)

Now draw boxes, enter FEMs values, rotation factors etc. As it is a two storeyed frame, calculations on a single A4 size paper may not be possible. A reduced page showing calculation is annexed.



Double – storey frame carrying gravity and lateral loads – Analysed by Rotation Contribution Method.

First Cycle :–

Near end contribution = Rotation factor of respective member (Restrained moment + far end contributions).

Joint B = R.F. (4.17)

C = R.F. (1.92 – 0.9)

D = R.F. (– 4.17 – 0.12)

E = R.F. (– 4.17 + 1.65)

After First Cycle :–

Linear Displacement Contribution :–= L.D.F.[Storey moment + Rotation contribution at the end of columns of that storey].

$L.D.C_1 = -0.75\ (-9 - 0.9 + 0.55) = 7$

$L.D.C_2 = -0.75\ (4.5 - 0.9 - 0.39 + 0.55 + 1.65) = 2.7$

For 2nd Cycle And Onwards :–

Near end contribution = R.F.[Restrained moment + Far end contribution + Linear displacement contributions of columns of different storeys meeting at that joint]

Joint B= R.F. (4.17 + 0.16 – 0.39 + 7 + 2.7)

C= ″ (1.92 + 0.49 – 2.96 + 2.7)

D= ″ (– 4.17 – 0.25 + 0.55 + 2.7)

E= ″ (– 4.17 + 0.45 – 0.89 + 2.7 + 7).

After 2nd Cycle :–

$L.D.C_1 = -0.75\ (-9 - 2.96 - 1.1) = 9.8$

$L.D.C_2 = -0.75\ (-4.5 - 2.96 - 0.83 - 1.1 + 0.45) = 6.71$

3rd Cycle :–

Joint B= R.F. (4.17 – 0.33 – 0.83 + 9.8 + 6.71)

C= ″ (1.92 + 0.13 – 4.24 + 6.71)

D= ″ (– 4.17 – 1.1 – 0.52 + 6.71)

E= ″ (– 4.17 – 1.27 – 0.35 + 9.8 + 6.71)

After 3rd Cycle :–

$L.D.C_1 = -0.75(-9 - 4.24 - 2.33) = 11.68$

$L.D.C_2 = -0.75(-4.5 - 1.74 - 4.24 - 0.35 - 2.33) = 9.87$

4th Cycle :–

Joint	B=	R.F. (4.17 – 0.70 – 1.74 + 11.68 + 9.87)
	C=	″ (1.92 – 0.11 – 5.05 + 9.87)
	D=	″ (– 4.17 – 0.76 – 2.33 + 9.87)
	E=	″ (– 4.17 – 1 – 1.51 + 9.87 + 11.68).

After 4th Cycle :–

$L.D.C_1 = -0.75(-9 - 5.05 - 3.23) = 12.96$

$L.D.C_2 = -0.75(-4.5 - 5.05 - 2.55 - 1.00 - 3.23) = 12.25$

5th Cycle :–

Joint	B=	R.F. (4.17 – 0.97 – 2.55 + 12.25 + 12.96)
	C=	″ (1.92 – 0.3 – 5.61 + 12.25)
	D=	″ (– 4.17 – 0.95 – 3.23 + 12.25)
	E=	″ (– 4.17 – 1.5 – 1.68 + 12.25 + 12.96)

After 5th Cycle :–

$L.D.C_1 = -0.75(-9 - 5.61 - 3.88) = 13.87$ (ground storey)

$L.D.C_2 = -0.75(-4.5 - 5.61 - 3.18 - 1.5 - 3.88) = 14$ (First Floor)

6th Cycle :–

Joint	B =	R.F. (4.17 – 1.16 – 3.18 + 14 + 13.87)
	C =	″ (1.92 – 0.05 – 6 + 14)
	D =	″ (– 4.17 – 3.88 – 1.09 + 14)
	E =	″ (– 4.17 – 1.87 – 1.68 + 14 + 13.87)

After 6th Cycle :–

$L.D.C_1 = -0.75\,(-9 - 6 - 4.37) = 14.53$

$L.D.C_2 = -0.75\,(-4.5 - 6 - 3.65 - 1.87 - 4.37) = 15.3$

7th Cycle :–

Joint	B	=	R.F. (4.17 – 1.31 – 3.65 + 15.3 + 14.53)
	C	=	″ (1.92 – 0.56 – 6.30 + 15.3)
	D	=	″ (– 4.17 – 1.19 – 4.37 + 15.3)
	E	=	″ (– 4.17 – 1.89 – 2.14 + 15.3 + 14.53)

After 7th Cycle :–

$L.D.C_1 = -0.75\,(-9 - 6.30 - 4.69) = 14.99$

$L.D.C_2 = -0.75\,(-4.5 - 6.3 - 3.99 - 2.14 - 4.69) = 16.21$

8th Cycle :–

Joint	B	=	R.F. (4.17 – 1.41 – 3.99 + 16.21 + 14.99)
	C	=	″ (1.92 – 6.5 – 0.64 + 16.21)
	D	=	″ (– 4.17 – 4.69 – 1.26 + 16.21)
	E	=	″ (– 4.17 – 2.34 – 1.95 + 16.21 + 14.99)

After 8th Cycle :–

$L.D.C_1 = -0.75\,(-9 - 6.5 - 4.93) \cong 15$

$L.D.C_2 = -0.75\,(-4.5 - 6.5 - 4.23 - 4.93 - 2.34) \cong 16.21$

FINAL END MOMENTS :–

(1) **Beams or Slabs :–**

= F.E.M + 2 (near end contribution) + far end contribution of that particular beam or slab.

(2) **For Columns :–**

= F.E.M + 2 (near end contribution) + far end contribution of that particular column + L.D.C. of that column. Applying these rules we get the following end moments.

END MOMENTS :–

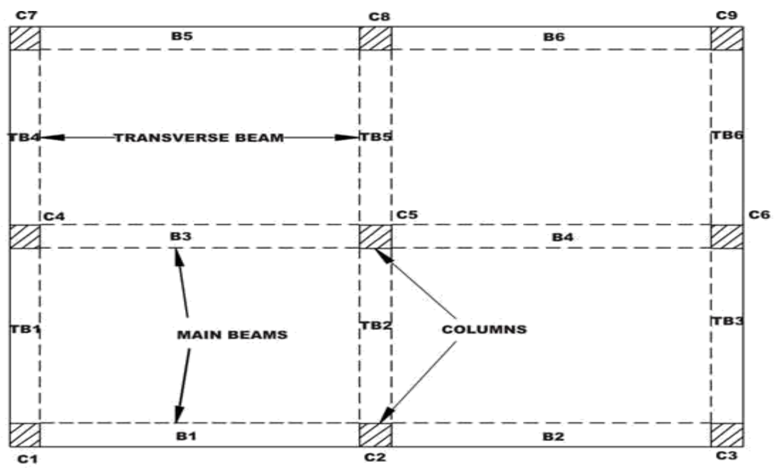
Mab =	2.25 + 2 × 0 – 6.5 + 15	=	+ 10.75 KN–m
Mba =	– 2.25 – 2 (6.5) – 1 + 15	=	– 0.25 ″
Mbc =	2.25 – 2 × 6.5 – 4.23 + 16.21	=	+ 1.23 ″
Mbe =	4.17 – 2 (1.95) – 1.48	=	– 1.21 ″
Mcb =	– 2.25 – 2 × 4.23 – 6.5 + 16.21	=	– 1 ″
Mcd =	4.17 – 2 × 1.26 – 0.7	=	+ 0.95≅+1 ″
Mdc =	– 4.17 – 2 × 0.7 – 1.26	=	– 6.83 ″
Mde =	0 – 2 × 2.34 – 4.93 + 16.21	=	+ 6.60 ″
Med =	0 – 2 × 4.93 – 2.34 + 16.21	=	+ 4.01 ″
Meb =	– 4.17 – 2 × 1.48 – 1.95	=	– 9.08 KN–m
Mef =	0 – 2 × 4.93 + 15	=	+ 5.14 ″
Mfe =	0 – 2 × 0 – 4.93 + 15	=	+ 10.07 ″

Now frame is statically determinate and contains all end moments. It can be designed now.

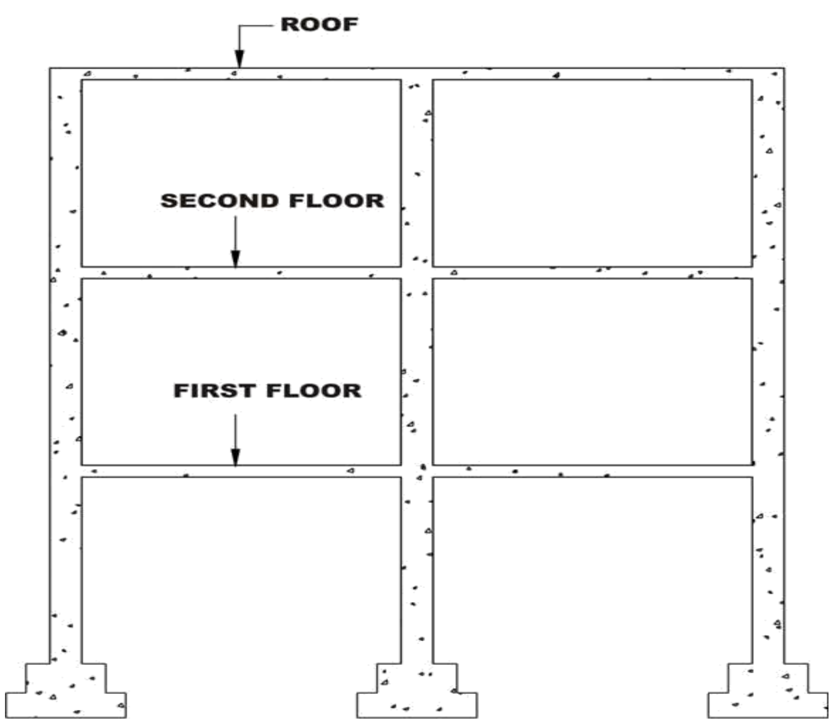
Space for notes:

36.1 Introduction

The building frames are the most common structural form, an analyst/engineer encounters in practice. Usually the building frames are designed such that the beam column joints are rigid. A typical example of building frame is the reinforced concrete multistory frames. A two-bay, three-storey building plan and sectional elevation are shown in Fig. 36.1. In principle this is a three dimensional frame.
However, analysis may be carried out by considering planar frame in two perpendicular directions separately for both vertical and horizontal loads as shown in Fig. 36.2 and finally superimposing moments appropriately. In the case of building frames, the beam column joints are monolithic and can resist bending moment, shear force and axial force. The frame has 12 joints (j), 15 beam members (b), and 9 reaction components (r).



Plan



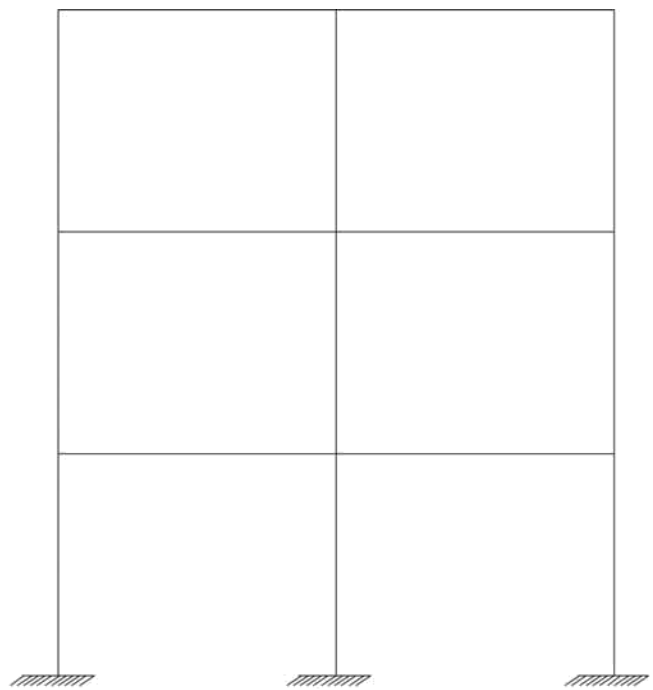


Fig.36.2 Idealized frame for analysis

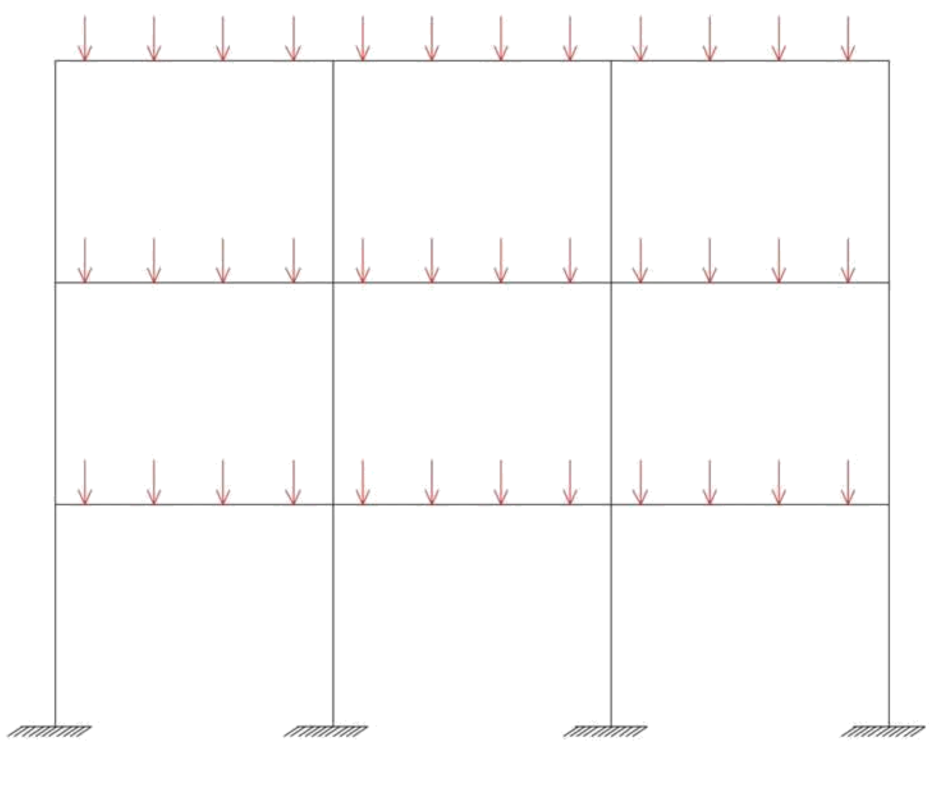


Fig.36.3 Building frame subjected to vertical loads

Analysis of Building Frames to Vertical Loads

Consider a building frame subjected to vertical loads as shown in Fig.36.3. Any typical beam, in this building frame is subjected to axial force, bending moment and shear force. Hence each beam is statically indeterminate to third degree and hence 3 assumptions are required to reduce this beam to determinate beam.

Before we discuss the required three assumptions consider a simply supported beam. In this case zero moment (or point of inflexion) occurs at the supports as shown in Fig.36.4a. Next consider a fixed-fixed beam, subjected to vertical loads as shown in Fig. 36.4b. In this case, the point of inflexion or point of zero moment occurs at $0.21L$ from both ends of the support.

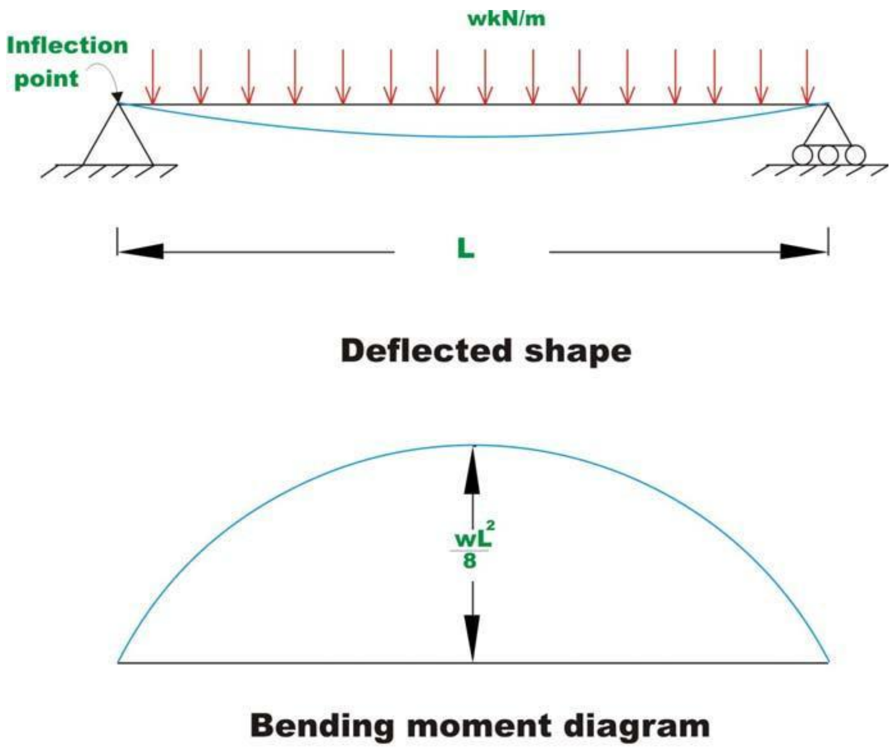
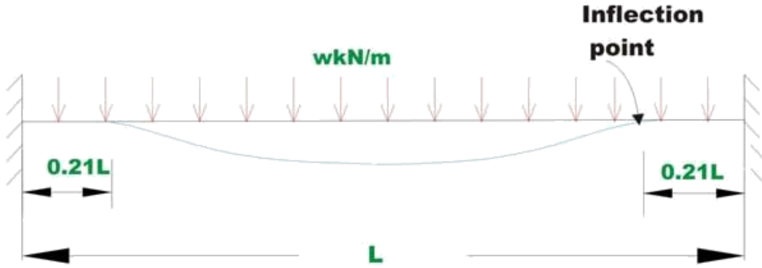
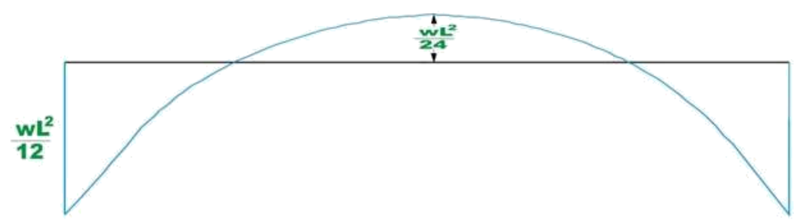


Fig.36. 4a Simply Supported beam



Deflected shape



Bending moment diagram

Fig.36. 4b Fixed - Fixed beam

Now consider a typical beam of a building frame as shown in Fig.36.4c. In this case, the support provided by the columns is neither fixed nor simply supported. For the purpose of approximate analysis the inflexion point or point of zero moment is assumed to occur at $\frac{0 + 0.21L}{2} \approx 0.1L$ from the supports. In reality the point of zero moment varies depending on the actual rigidity provided by the columns. Thus the beam is approximated for the analysis as shown in Fig.36.4d.

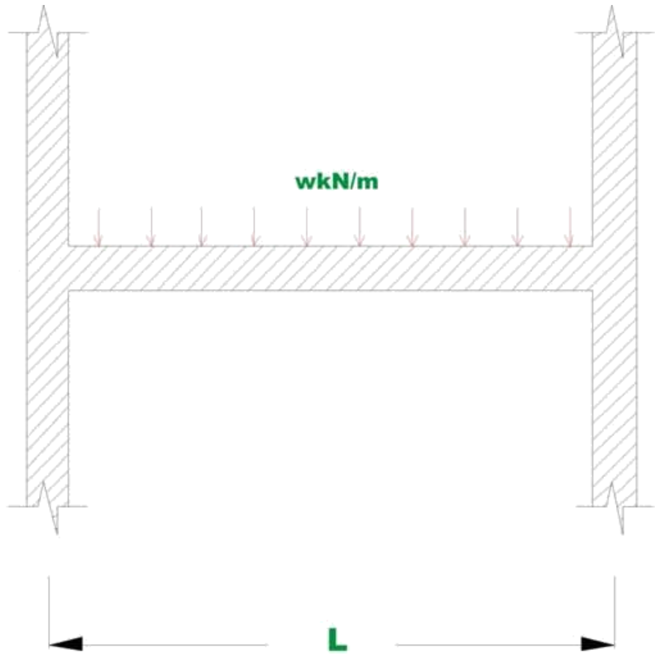
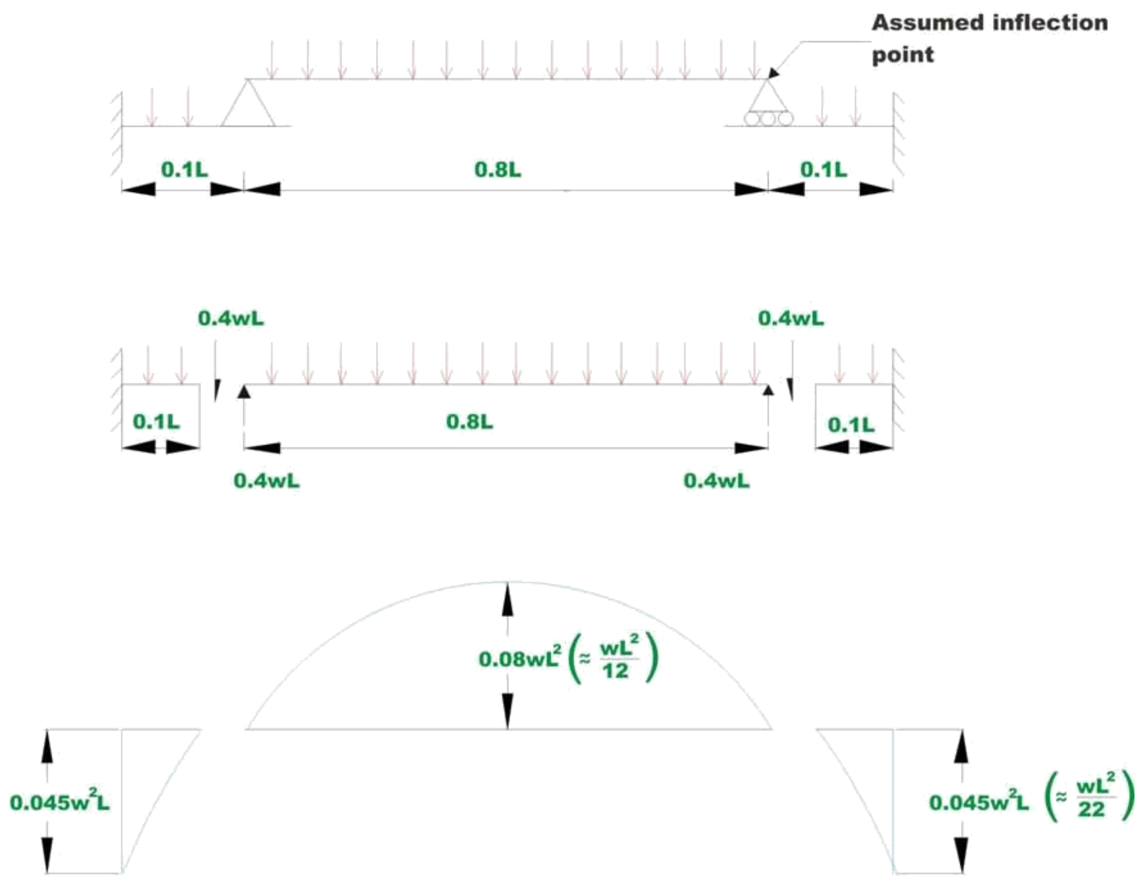


Fig.36.4c



Bending moment diagram

Fig.36.4d

For interior beams, the point of inflexion will be slightly more than $0.1L$. An experienced engineer will use his past experience to place the points of inflexion appropriately. Now redundancy has reduced by two for each beam. The third assumption is that axial force in the beams is zero. With these three assumptions one could analyse this frame for vertical loads.

Example 1

Analyse the building frame shown in Fig. 36.5a for vertical loads using approximate methods.

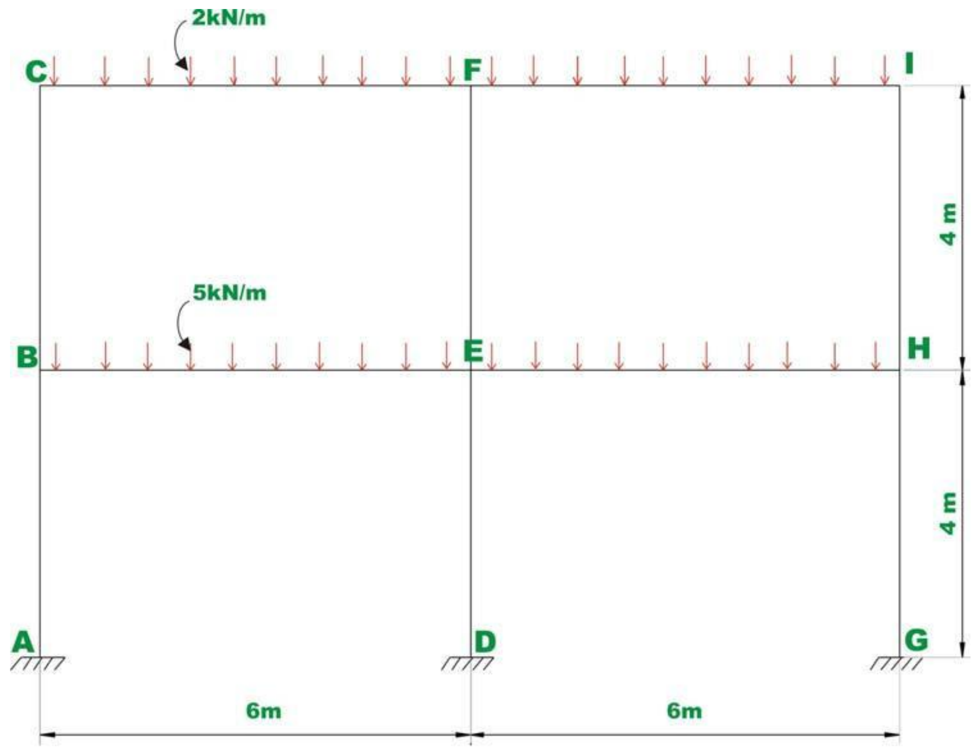


Fig.36.5a

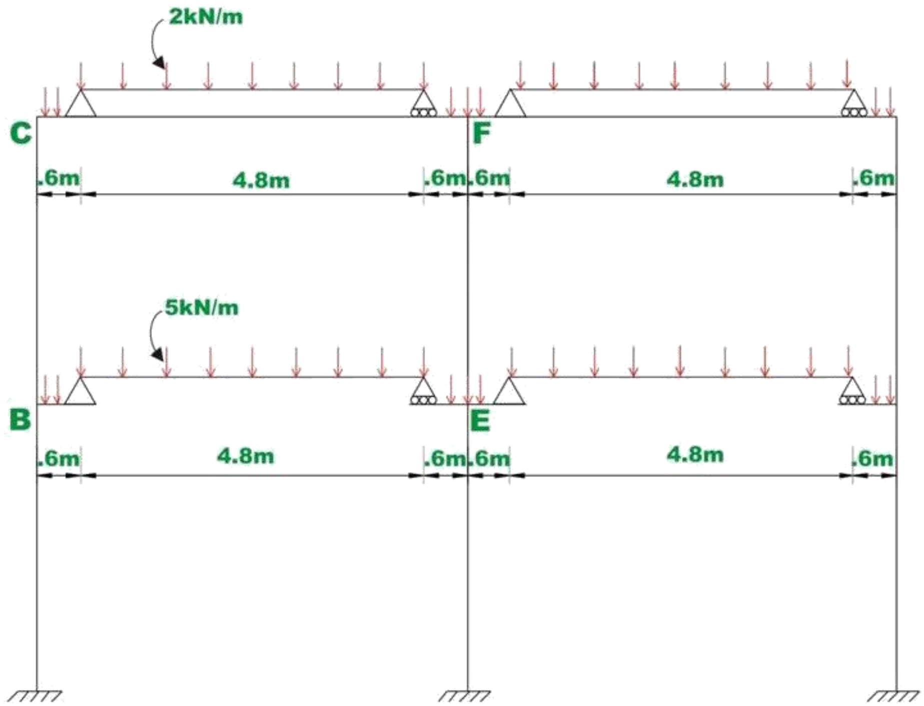
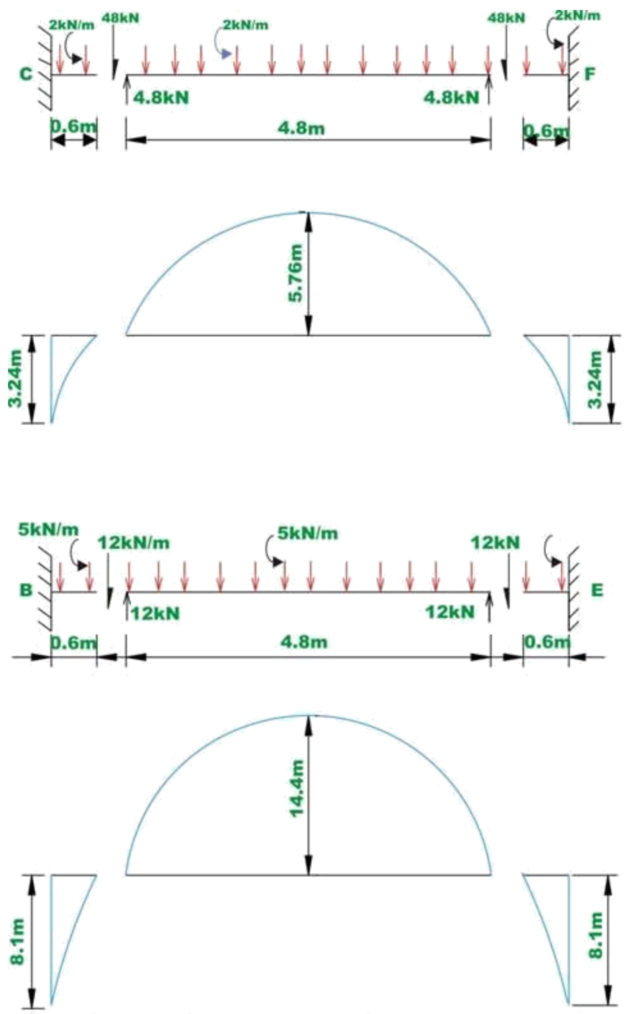


Fig.36.5 b

Solution:

In this case the inflexion points are assumed to occur in the beam at $0.1L(=0.6m)$ from columns as shown in Fig. 36.5b. The calculation of beam moments is shown in Fig. 36.5c.



Bending moment diagrams

Fig.36.5c

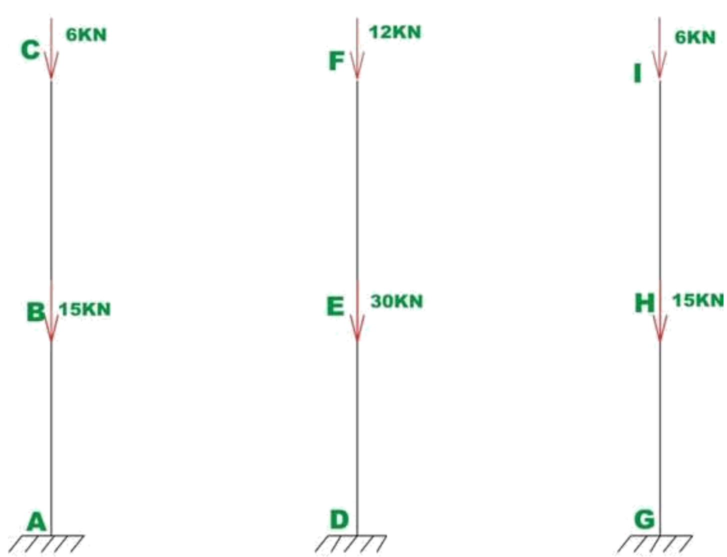


Fig.36.5d Axial force in columns

Now the beam $-ve$ moment is divided equally between lower column and upper column. It is observed that the middle column is not subjected to any moment, as the moment from the right and the moment from the left column balance each other. The $-ve$ moment in the beam BE is 8.1kN.m. Hence this moment is divided between column BC and BA. Hence, $M_{BC} = M_{BA} = \frac{8.1}{2} = 4.05 \text{kN.m}$. The maximum $+ve$ moment in beam BE is 14.4 kN.m. The columns do carry axial loads. The axial compressive loads in the columns can be easily computed. This is shown in Fig. 36.5d.

Analysis of Building Frames to lateral (horizontal) Loads

A building frame may be subjected to wind and earthquake loads during its life time. Thus, the building frames must be designed to withstand lateral loads. A two-storey two-bay multistory frame subjected to lateral loads is shown in Fig. 36.6. The actual deflected shape (as obtained by exact methods) of the frame is also shown in the figure by dotted lines. The given frame is statically indeterminate to degree 12.

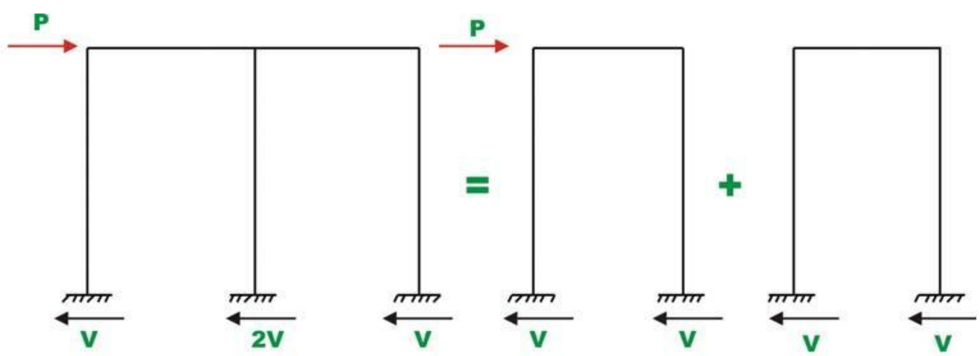


Fig.36.6 Shear in columns

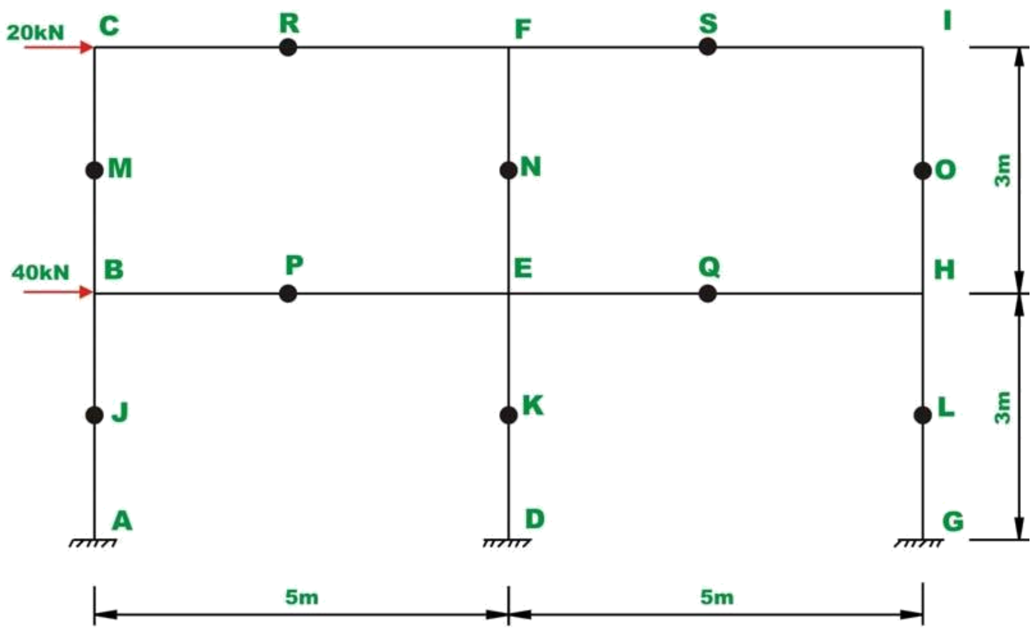


Fig.36.7a Two storey building frame subjected to lateral load of Example 36.2

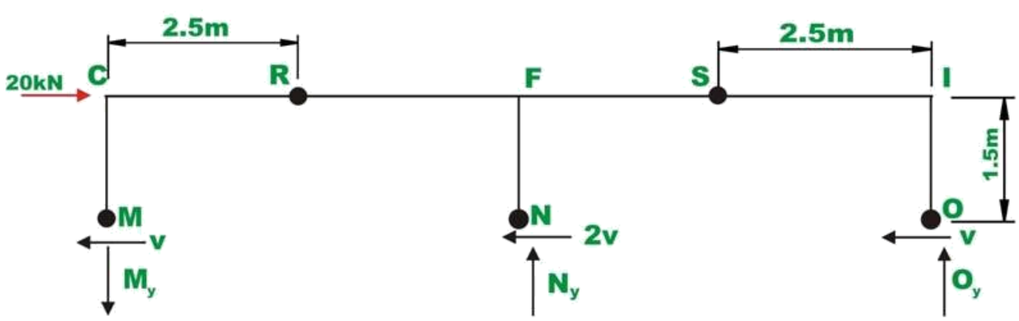


Fig.36.7b

Hence it is required to make 12 assumptions to reduce the frame in to a statically determinate structure. From the deformed shape of the frame, it is observed that inflexion point (point of zero moment) occur at mid height of each column and mid point of each beam. This leads to 10 assumptions. Depending upon how the remaining two assumptions are made, we have two different methods of analysis: *i*) Portal method and *ii*) cantilever method. They will be discussed in the subsequent sections.

Portal method

In this method following assumptions are made.

1) An inflexion point occurs at the mid height of each column.
2) An inflexion point occurs at the mid point of each girder.

3) The total horizontal shear at each storey is divided between the columns of that storey such that the interior column carries twice the shear of exterior column.
The last assumption is clear, if we assume that each bay is made up of a portal thus the interior column is composed of two columns (Fig. 36.6). Thus the interior column carries twice the shear of exterior column. This method is illustrated in example 36.2.

Example 3

Analyse the frame shown in Fig. 36.7a and evaluate approximately the column end moments, beam end moments and reactions.

Solution:
The problem is solved by equations of statics with the help of assumptions made in the portal method. In this method we have hinges/inflexion points at mid height of columns and beams. Taking the section through column hinges $M .N , O$ we get, (ref. Fig. 36.7b).

$$\sum F_X = 0 \quad \Rightarrow \quad V + 2V + V = 20$$

or $V = 5$ kN
Taking moment of all forces left of hinge R about R gives,

$$V \times 1.5 - M_y \times 2.5 = 0$$
$$M_y = 3 \text{ kN}(\downarrow)$$

Column and beam moments are calculates as,

$$M_{CB} = 5 \times 1.5 = 7.5 \text{ kN.m} \; ; M_{IH} = +7.5 \text{ kN.m}$$

$$M_{CF} = -7.5 \text{ kN.m}$$

Taking moment of all forces left of hinge S about S gives,

$$5 \times 1.5 - O_y \times 2.5 = 0$$

$$O_y = 3\text{kN}(\uparrow)$$

$$N_y = 0$$

Taking a section through column hinges J , K, L we get, (ref. Fig. 36.7c).

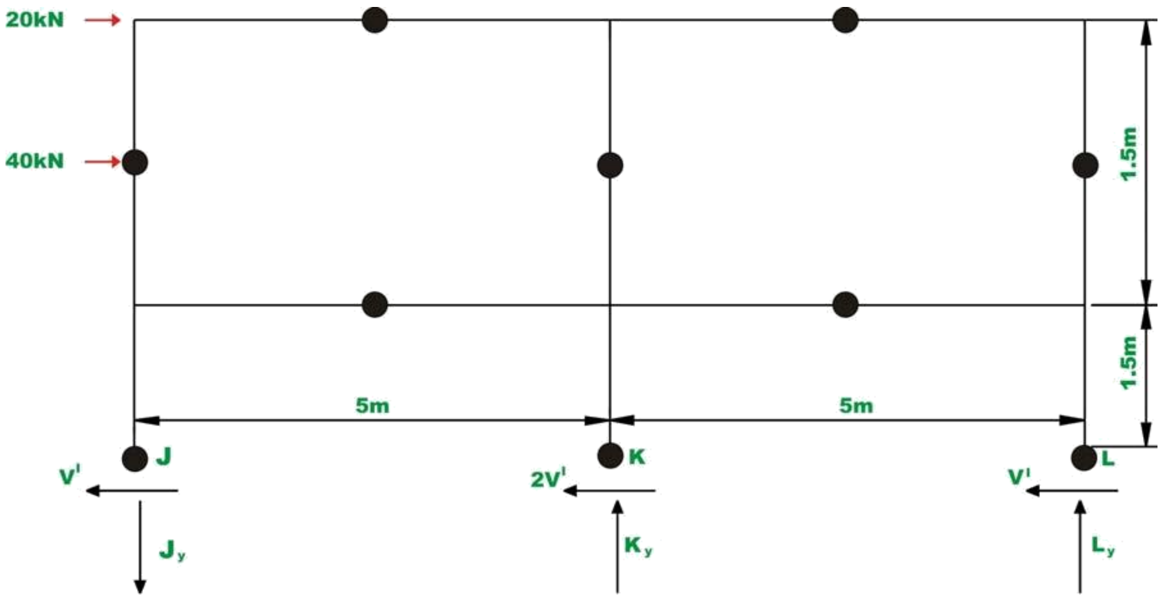


Fig.36.7c

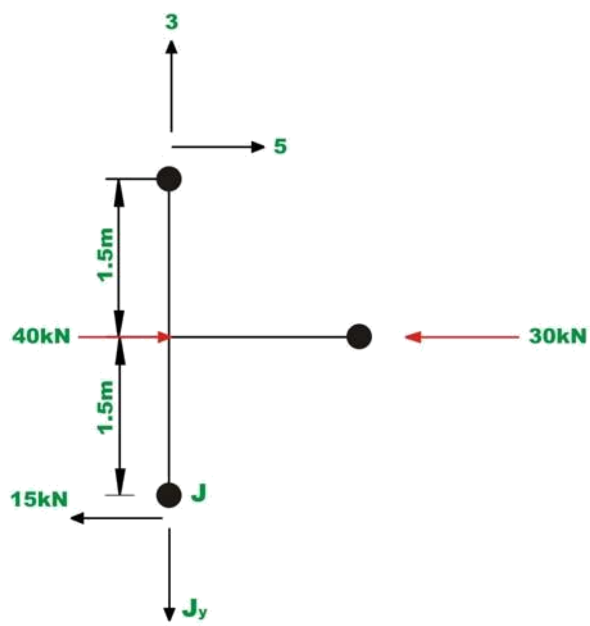


Fig.36.7d

$\sum F_X = 0 \quad \Rightarrow \quad V' + 2V' + V' = 60$

or $V' = 15$ kN

Taking moment of all forces about P gives (vide Fig. 36.7d)

$$\sum M_p = 015 \times 1.5 + 5 \times 1.5 + 3 \times 2.5 - J_y \times 2.5 = 0$$

$$J_y = 15 \text{ kN} (\downarrow)$$

$$L_y = 15 \text{ kN} (\uparrow)$$

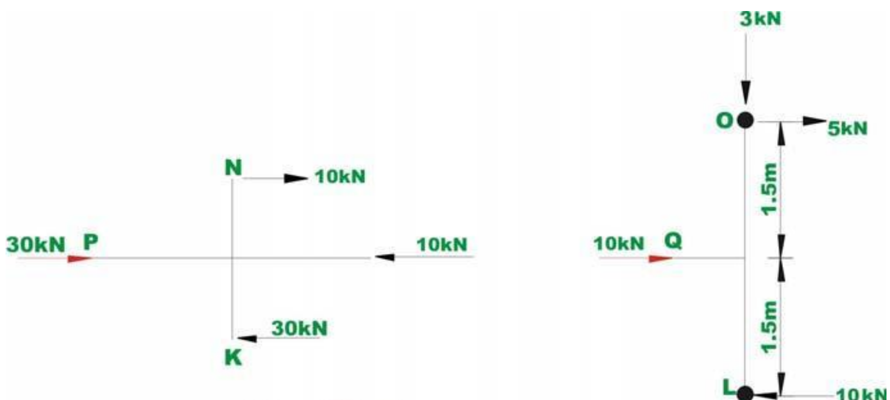


Fig.36.7e

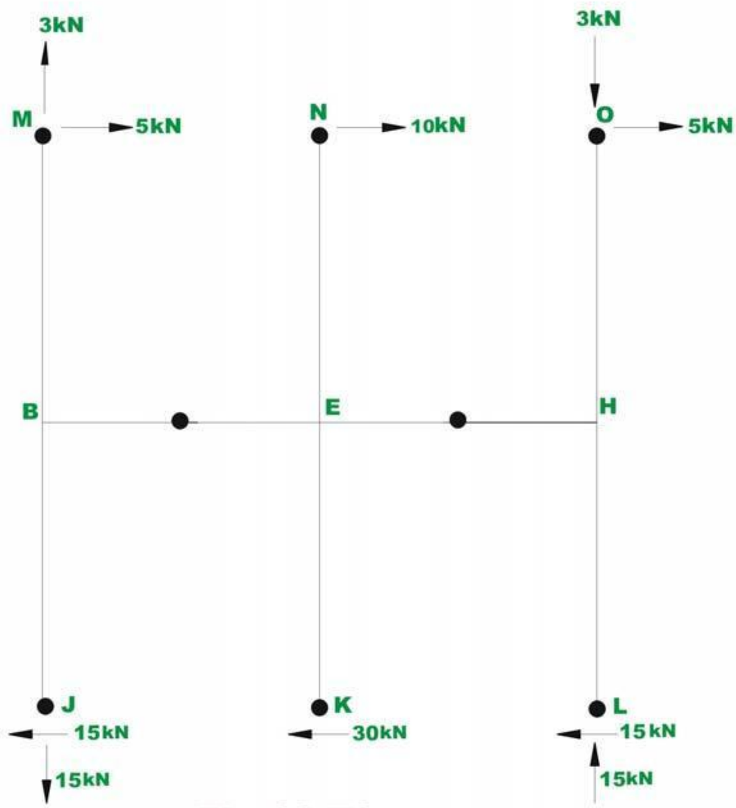


Fig.36.7f

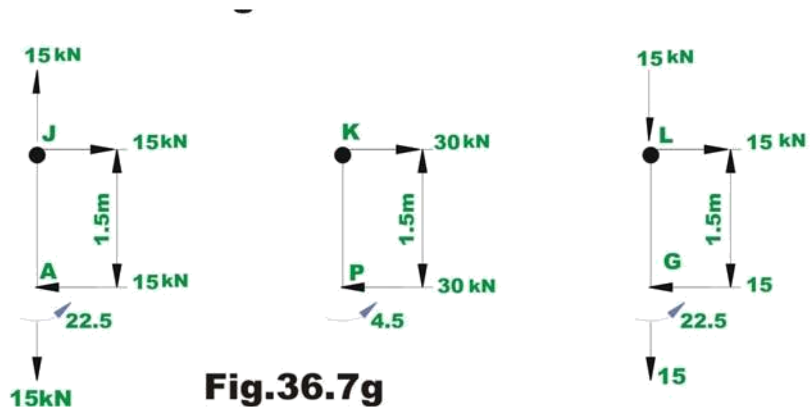


Fig.36.7g

Column and beam moments are calculated as, (ref. Fig. 36.7f)

$M_{BC} = 5 \times 1.5 = 7.5\ kN.m$; $M_{BA} = 15 \times 1.5 = 22.5$ kN.m

$M_{BE} = -30$ kN.m

$M_{EF} = 10 \times 1.5 = 15$ kN.m ; $M_{ED} = 30 \times 1.5 = 45$ kN.m

$M_{EB} = -30$ kN.m $M_{EH} = -30$ kN.m

$M_{HI} = 5 \times 1.5 = 7.5$ kN.m ; $M_{HG} = 15 \times 1.5 = 22.5$ kN.m

$M_{HE} = -30$ kN.m

Reactions at the base of the column are shown in Fig. 36.7g.

Cantilever method

The cantilever method is suitable if the frame is tall and slender. In the cantilever method following assumptions are made.

1) An inflexion point occurs at the mid point of each girder.
2) An inflexion point occurs at mid height of each column.
3) In a storey, the intensity of axial stress in a column is proportional to its horizontal distance from the center of gravity of all the columns in that storey.

Consider a cantilever beam acted by a horizontal load P as shown in Fig. 36.8. In such a column the bending stress in the column cross section varies linearly from its neutral axis. The last assumption in the cantilever method is based on this fact. The method is illustrated in example 36.3.

Example 4

Estimate approximate column reactions, beam and column moments using cantilever method of the frame shown in Fig. 36.8a. The columns are assumed to have equal cross sectional areas.

Solution:

This problem is already solved by portal method. The center of gravity of all column passes through centre column.

$$\bar{x} = \frac{\sum xA}{\sum A} = \frac{(0)A + 5A + 10A}{A + A + A} = 5 \text{ m (from left column)}$$

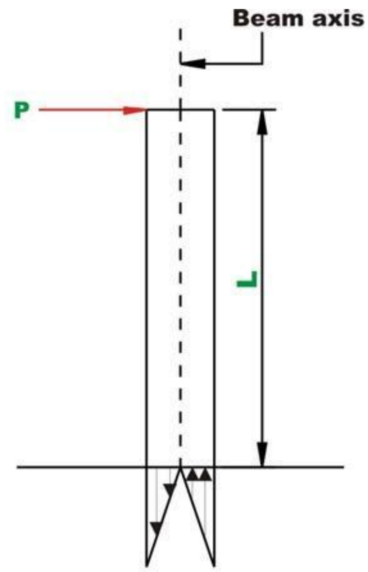


Fig.36.8a Cantilever Column

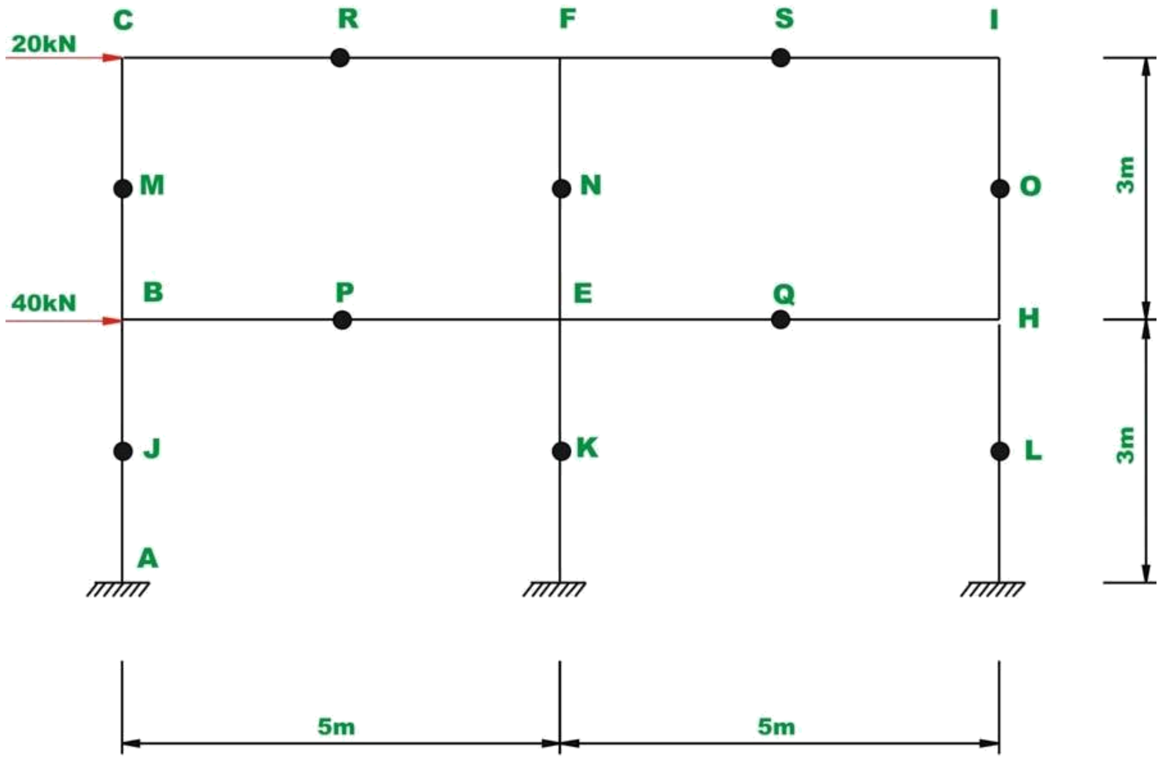


Fig.36.8b

Taking a section through first storey hinges gives us the free body diagram as shown in Fig. 36.8b. Now the column left of C.G. *i.e.* CB must be subjected to tension and one on the right is subjected to compression.

From the third assumption,

$$\frac{M_y}{5 \times A} = -\frac{O_y}{5 \times A} \qquad \Rightarrow M_y = -O_y$$

Taking moment about O of all forces gives,

$$20 \times 1.5 - M_y \times 10 = 0$$

$$M_y = 3\text{kN}(\downarrow); \qquad O_y = 3\text{ kN}(\uparrow)$$

Taking moment about R of all forces left of R ,

$$V_M \times 1.5 - 3 \times 2.5 = 0$$

$$V_M = 5 \text{ kN} (\leftarrow)$$

Taking moment of all forces right of S about S ,

$$V_O \times 1.5 - 3 \times 2.5 = 0 \Rightarrow V_O = 5 \text{ kN.}$$

$$\sum F_X = 0 \qquad V_M + V_N + V_O - 20 = 0$$

$$V_N = 10 \text{ kN.}$$

Moments

$$M_{CB} = 5 \times 1.5 = 7.5 \text{ kN.m}$$

$$M_{CF} = -7.5 \text{ kN.m}$$

$$M_{FE} = 15 \text{ kN.m}$$

$$M_{FC} = -7.5 \text{ kN.m}$$

$$M_{FI} = -7.5 \text{ kN.m}$$

$$M_{IH} = 7.5 \text{ kN.m}$$

$$M_{IF} = -7.5 \text{ kN.m}$$

Tae a section through hinges J , K, L (ref. Fig. 36.8c). Since the center of gravity passes through centre column the axial force in that column is zero.

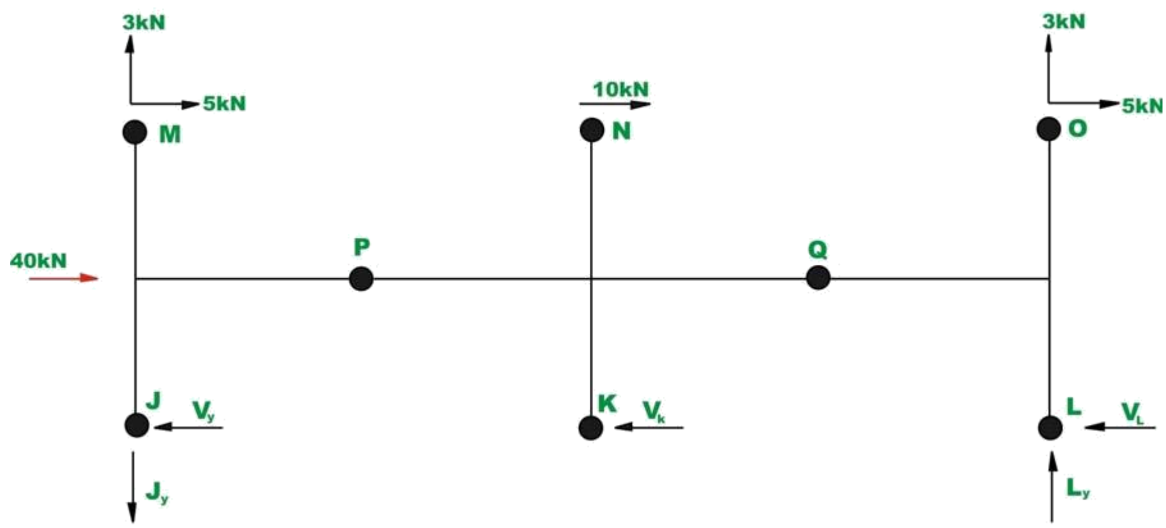


Fig.36.8c

Taking moment about hinge L , J_y can be evaluated. Thus,

$$20 \times 3 + 40 \times 1.5 + 3 \times 10 - J_y \times 10 = 0$$

$$J_y = 15\text{kN}(\downarrow) \; ; \quad L_y = 15\text{kN}(\uparrow)$$

Taking moment of all forces left of P about P gives,

$$5 \times 1.5 + 3 \times 2.5 - 15 \times 2.5 + V_j \times 1.5 = 0$$

$$V_J = 15 \text{ kN}(\leftarrow)$$

Similarly taking moment of all forces right of Q about Q gives,

$$5 \times 1.5 + 3 \times 2.5 - 15 \times 2.5 + V_L \times 1.5 = 0$$

$$V_L = 15\text{kN}(\leftarrow)$$

$$\sum F_X = 0 \qquad V_J + V_K + V_L - 60 = 0$$

$$V_K = 30 \text{ kN.}$$

Moments

$$M_{BC} = 5 \times 1.5 = 7.5 \text{ kN.m} \quad ; \quad M_{BA} = 15 \times 1.5 = 22.5 \text{ kN.m}$$

$$M_{BE} = -30 \text{ kN.m}$$

$$M_{EF} = 10 \times 1.5 = 15 \text{ kN.m} \quad ; \quad M_{ED} = 30 \times 1.5 = 45 \text{ kN.m}$$

$$M_{EB} = -30 \text{ kN.m} \qquad M_{EH} = -30 \text{ kN.m}$$

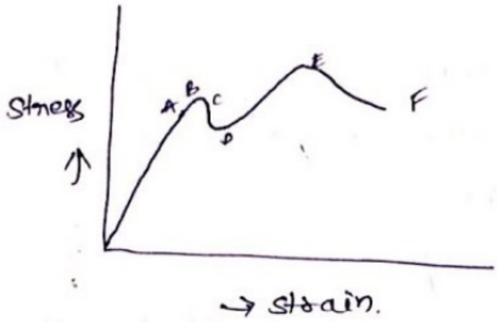
$$M_{HI} = 5 \times 1.5 = 7.5 \text{ kN.m} \quad ; \quad M_{HG} = 15 \times 1.5 = 22.5 \text{ kN.m}$$

$$M_{HE} = -30 \text{ kN.m}$$

UNIT-VI
PLASTIC ANALYSIS

Introduction:

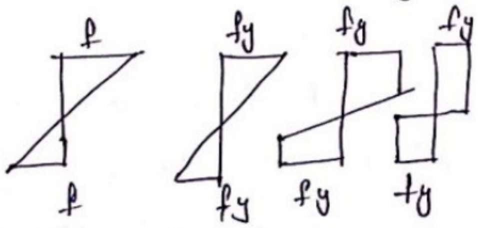
Stress strain relation is assumed to be linear in elastic theory. The design based on this theory assumes that the structure fails if the structure @ any point reaches the yield stress. The working stress is defined as yield stress divided by F.S. That means the structure would fail if the design load applied is equal to F.S times the working load. But this is not correct concept. To verify this, first let us see stress strain curve for steel



A – Proportional limit
B – Elastic limit
C – upper yield
D – lower yield
E – ultimate limit
F – Breaking limit.

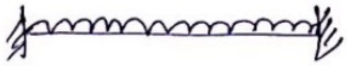
Now, consider stresses across highly stressed section of SSB with gradually (↑) load.

i) Beam subjected to gradual (↑) Load.



ii) stress diagram @ various loading stages.

→ Let us consider load carrying capacity of fixed beam



As B.M is max @ supports first extreme fibres @ supports yield. For further (↑) of load, entire section @ supports yield.

Thus elastic theory underestimates the load carrying capacity of structure.

Hence a new theory is developed and it named as "PLASTIC THEORY".

→ It gives correct idea about the Load carrying capacity of the Structure

→ It is based on concept that a structure will carry load till the Plastic hinges are formed @ sufficient points to cause collapse of the structure

PLASTIC HINGE

Plastic hinge is a section in which all fibres yield, and hence for any further load rotation takes place @ the section without resisting any additional moment.

PLASTIC MOMENT CAPACITY

Plastic moment capacity of a section is defined as moment which makes all the fibres @ that section to yield & thereby form a plastic Hinge.

Assumptions in plastic theory

1) Stress - strain relationship is idealized to two straight lines.

~~elastic~~ strain hardening effect is neglected.

2) plane section remains plane before & after bending ie; shear defo is neglected

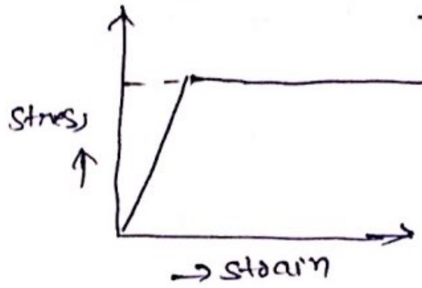
3) The relationship b/w comp. stress & comp. strain is same as tensile stress & tensile strain

4) Whenever a fully plastic moment is attained @ any c/s, a plastic hinge forms which may undergo rotation of any magnitude, but the B.M remains constant @ the fully plastic value.

5) Effect of axial load & shear on fully plastic moment capacity of section is neglected

6) The deflections in the structure are small, enough for equations of statical equilibrium to be same as those for undeformed structures.

Idealized stress strain curve : The stress-strain relationship is "idealized" to two st-lines ie; strain-hardening effect is neglected.



(3)

Shape factor

shape factor defined as ratio of plastic moment capacity to the yield moment. It is denoted by 'S'.

$$S = \frac{Mp}{My}.$$

Mp → plastic moment capacity : $Mp = f_y z_p$

where z_p = plastic modulus of section.

f_y = ~~characteristic comp. strength~~. yield strength

My → yield moment : $My = f_y z$.

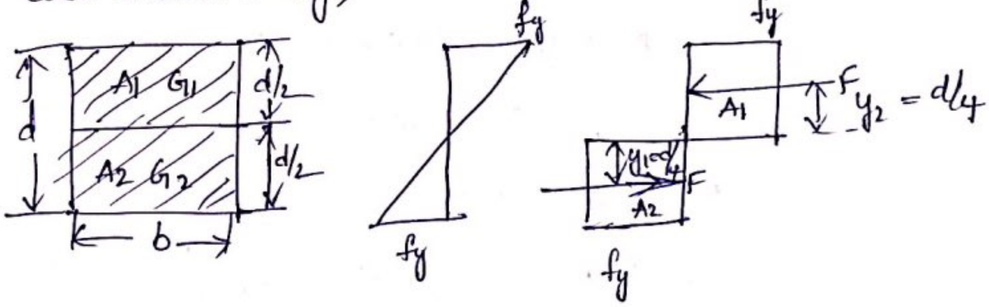
z = Modulus of section.

$$\therefore S = \frac{f_y z_p}{f_y z} \quad (\text{or}) \quad \boxed{S = \frac{z_p}{z}}$$

∴ Shape factor is always > 1 [∴ Mp is more than My]

Shape factors for various sections

a) Rectangular section : consider rectangular section of width 'b' and depth 'd' as shown in fig. The stress diag. corresponding to yield moment and plastic moment also shown in fig;



We know that yield moment $My = f_y z$

$$My = f_y\left(\frac{bd^2}{6}\right) \quad \text{--- (1)}$$

Let A1 be area under compression and A2 be area under tension. (4)

Considering Horizontal equilibrium of forces

$$F_c = F_t$$

$$f_y A_1 = f_y A_2 \Rightarrow A_1 = A_2 = \frac{A}{2}$$ where A is total area

Plastic moment capacity is the moment of resistance when the yield stress is f_y @ all fibres.

$$M_p = F_c y_1 + F_t y_2$$

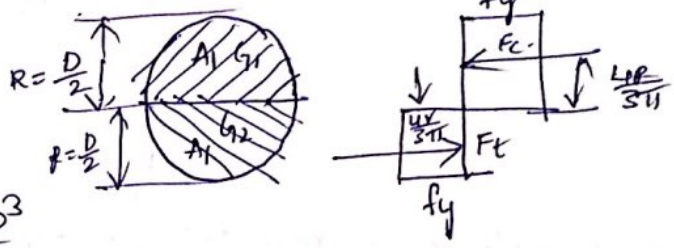
$$= f_y A_1 y_1 + f_y A_2 y_2$$

$$= f_y \times \frac{A}{2}(y_1 + y_2)$$

$$= \left(f_y \times \frac{bd}{2}\right)\left(\frac{d}{4} + \frac{d}{4}\right) \quad \therefore \boxed{M_p = f_y \frac{bd^2}{4}}$$

$$\text{Shape factor} = \frac{M_p}{M_y} = \frac{f_y \frac{bd^2}{4}}{f_y \frac{bd^2}{6}} = 1.5$$

b) Circular section: consider circular section of radius R. Let the dia. be denoted by D.



$$I = \frac{\pi D^4}{64} \; ; \; y_{max} = \frac{D}{2}.$$

$$Z = \frac{I}{y_{max}} = \frac{\pi D^4}{64} \times \frac{2}{D} = \frac{\pi D^3}{32}.$$

C.G of comp. area $y_1 = \frac{4R}{3\pi}$ above diametrical section

C.G of tensile area $y_2 = \frac{4R}{3\pi}$ below " "

$$M_p = F_c y_1 + F_t y_2$$

$$= f_y A_1 y_1 + f_y A_2 y_2$$

$$= f_y \times \frac{A}{2}(y_1 + y_2)$$

But $A = \pi D^2/4$

$$\therefore M_p = \frac{1}{2} \times \frac{\pi D^2}{4} \times f_y \left(\frac{4R}{3\pi} + \frac{4R}{3\pi}\right)$$

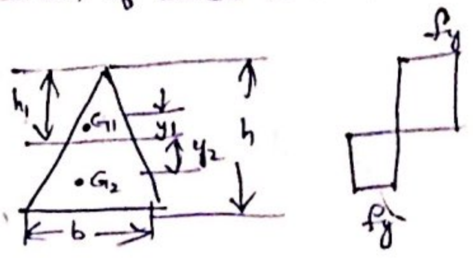
$$= \frac{1}{2} \times \frac{\pi D^2}{4} \times f_y \left(\frac{4D}{3\pi}\right)$$

$$M_p = f_y \frac{D^3}{6}.$$

$$S = \frac{M_p}{M_y} = \frac{f_y D^3/6}{f_y \pi D^3/32} = \frac{16}{\pi} = 1.698.$$

(5)

c) Triangular section. Consider typical triangular section of base width 'b' & depth 'h' as shown in fig.,



$$I = \frac{bh^3}{36} ; \quad y_{max} = \frac{2}{3}h$$

$$M_y = f_y z = f_y \left(\frac{I}{y_{max}}\right)$$
$$= f_y \left(\frac{bh^3/36}{2/3\, h}\right) \quad \therefore \quad M_y = f_y \left(\frac{bh^2}{24}\right) \quad \text{--- (1)}$$

Let the NA be @ a depth h_1 from apex. @ this level, width $b_1 = \left(\frac{h_1}{h}\right) b$.

Area under compression $= \frac{1}{2} b_1 h_1 = \frac{1}{2}\left(\frac{h_1}{h}\right) b\, h_1 = \frac{1}{2}\frac{h_1^2 b}{h}$

$$= \frac{A}{2} = \frac{1}{2}\left(\frac{bh}{2}\right)$$

$$\Rightarrow \frac{1}{2}\frac{h_1^2 b}{h} = \frac{1}{4} bh \quad \Rightarrow \quad \boxed{h_1 = \frac{h}{\sqrt{2}}}$$

$$b_1 = \frac{h_1}{h} b = \frac{h}{\sqrt{2}h} b = \frac{b}{\sqrt{2}}.$$

Distance of centroid compression area from plastic N.A

$$y_1 = \frac{h_1}{3} = \frac{h}{3\sqrt{2}}.$$

Distance of centroid of tensile area from plastic NA

$$y_2 = \left(\frac{h - h_1}{3}\right) \times \left(\frac{b_1 + 2b}{b_1 + b}\right)$$

$$= \left(\frac{h - (h/\sqrt{2})}{3}\right) \times \left(\frac{(b/\sqrt{2}) + 2b}{(b/\sqrt{2}) + b}\right)$$

$$= h\left(\frac{\sqrt{2}-1}{3\sqrt{2}}\right) \times \left(\frac{1+2\sqrt{2}}{1+\sqrt{2}}\right) = 0.1548h.$$

$$M_p = F_y A_1 y_1 + f_y A_2 y_2$$

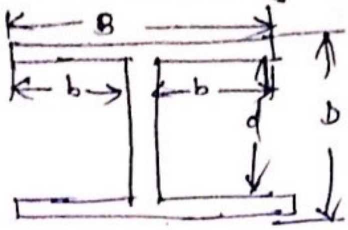
$$= f_y \frac{A}{2}(y_1 + y_2)$$

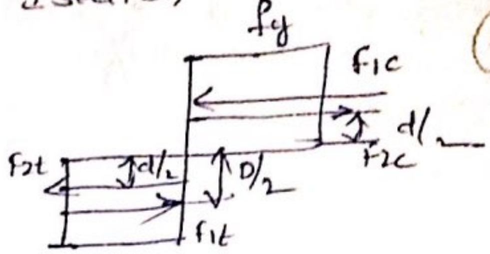
$$= f_y \times \frac{1}{2}\frac{bh}{2}\left(\frac{h}{3\sqrt{2}} + 0.1548h\right)$$

$$= 0.09763\, bh^2 f_y$$

Shape factor $S = \frac{M_p}{M_y} = 0.09763\left[\frac{bh^2 f_y}{f_y\left(\frac{bh^2}{24}\right)}\right] = 2.343$

d) Symmetric I Section Consider typical I section





(6)

$$\therefore M_y = f_y \left[\frac{\left(\frac{(BD)^3}{12}\right) - \left(\frac{(bd)^3}{12}\right)}{\left(\frac{D}{2}\right)} \right] = \frac{1}{6} f_y \left(\frac{BD^3 - bd^3}{D} \right)$$

Consider solid rectangle of size $B \times D$ & a negative area of $b \times d$.

$$M_p = F_{c1} \times \frac{D}{4} + F_{t1} \times \frac{D}{4} - F_{c2} \times \frac{d}{4} - F_{t2} \times \frac{d}{4}$$

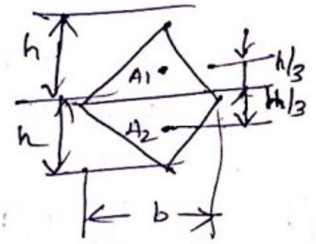
$$= f_y \left[\left(\frac{BD}{2} \times \frac{D}{4} \right) + \left(\frac{BD}{2} \times \frac{D}{4} \right) - \left(\frac{bd}{2} \times \frac{d}{4} \right) - \left(\frac{bd}{2} \times \frac{d}{4} \right) \right]$$

$$= f_y \left[\frac{BD^2}{4} - \frac{bd^2}{4} \right]$$

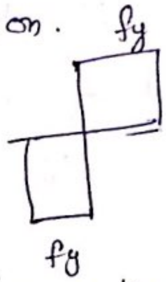
$$\therefore M_p = \frac{f_y}{4} (BD^2 - bd^2)$$

Shape factor $S = \frac{M_p}{M_y} = 1.5 \left[\frac{(BD^2 - bd^2) D}{(BD^3 - bd^3)} \right]$

e) for Diamond section Consider diamond section.



Diamond section



Stress diagram corresponding to plastic moment

$$I = 2 \times \frac{1}{12} bh^3 = \frac{1}{6} bh^3$$

$$M_y = f_y \left(\frac{I}{y_{max}} \right) = f_y \left(\frac{\frac{1}{6} bh^3}{h} \right) = f_y \frac{1}{6} bh^2$$

$$A_1 = A_2 = \frac{1}{2} bh \quad , y_1 = y_2 = \frac{1}{3} h$$

$$M_p = F_c y_1 + F_t y_2$$

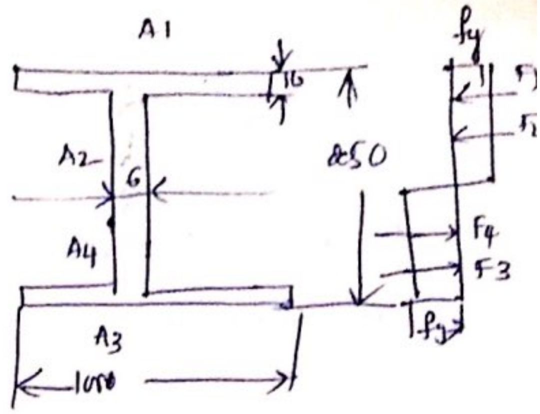
$$= f_y A_1 y_1 + f_y A_2 y_2$$

$$= f_y \times \frac{1}{2} bh \times \frac{1}{3} h + f_y \times \frac{1}{2} bh \times \frac{h}{3} = f_y \left(\frac{bh^2}{3} \right)$$

$$S = \frac{M_p}{M_y} = \frac{f_y \left(\frac{bh^2}{3} \right)}{f_y \left(\frac{bh^2}{6} \right)} \qquad \therefore S = 2.0$$

Q1) Calculate shape factor of I section shown in fig. If the value of yield stress is 250N/mm², find plastic moment capacity of section. (7)

Sol $S = \frac{M_p}{M_y}$



My yield moment

Due to symmetry, N.A is @ mid depth

$$I = \frac{100\times10^3}{12} + (100\times10)(125-5)^2 + \frac{6\times230^3}{12} + \frac{100\times10^3}{12} + (100\times10)(125-5)^2$$

$$I = 34900167\,mm^4$$

$$M_y = f_y\left(\frac{I}{y_{max}}\right) = 250\left(\frac{34900167}{125}\right) = 69800333\,Nmm$$

plastic moment (M_p) Since axis @ mid depth is a symmetric axes, plastic N.A also lies here. Dividing the given fig. into four simple rectangles, two in compression & two in tension zone.

$$M_p = \Sigma A f_y y$$

$A_1 = A_3$

$A_2 = A_4$.

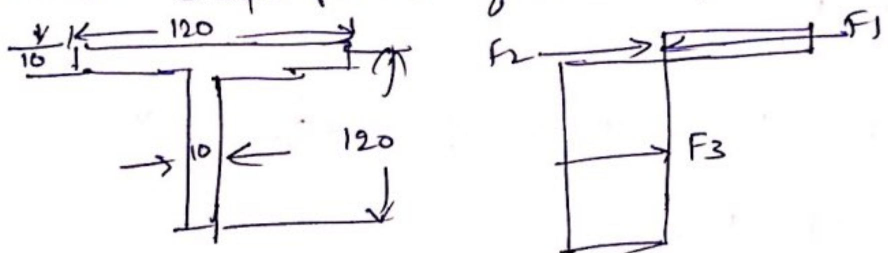
$$= f_y(A_1y_1 + A_2y_2 + A_3y_3 + A_4y_4)$$

$$= f_y\big(100\times10(125-5) + 6(125-10) + 100\times10(125-5) + 6(125-10)\big)$$

$$M_p = 79837500\,Nmm$$

$$S = \frac{M_p}{M_y} = 1.144$$

② Determine shape factor of T-section



Sol M_y Given fig. divided into '2' rectangles

N.A from top fibre

$$y_t = \frac{120\times10\times5 + (120-10)\times10(55+10)}{120\times10 + 110\times10}$$

$$y_t = 33.7\,mm$$

$$I = \frac{120\times10^3}{12} + 120\times10\times(33.7-5)^2 + \frac{10\times110^3}{12} + 10\times110\times(65-33.70)$$

(8)

$I = 3185523.7\,mm^4$

$y_{max} = 120 - 33.70 = 86.3\,mm$

$$M_y = f_y\left(\frac{I}{y_{max}}\right) = f_y\left(\frac{3185523.7}{86.3}\right) = 36909.68\,f_y\ Nmm.$$

Mp Assuming plastic N.A lies in flange, its distance from top of fibre, y_p can be found as

$$y_p \times 120 = \frac{A}{2} = \frac{1}{2}(120\times110 + 110\times10)$$

$y_p = 9.583\,m\ (<10\,mm)$ Hence assump. correct

Area under compression $A_1 = 120\times9.583$; $y_1 = \frac{9.583}{2}$

" tension $A_2 = 120\times(10-9.583)$

$A_2 = 120\times0.417$; $y_2 = \frac{0.417}{2}$

$A_3 = 10\times110 = 1100\,mm^2$; $y_3 = 0.417 + \frac{110}{2}$

$y_3 = 55.41\,mm$

$\therefore M_p = f_y \Sigma Ay$

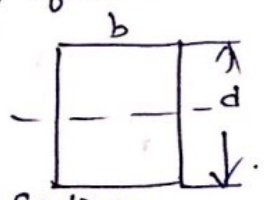
$= f_y\left[A_1y_1 + A_2y_2 + A_3y_3\right]$

$M_p = 66479.17\,f_y\ Nmm$ $\therefore S = \frac{M_p}{M_y} = 1.801$

Moment curvature relationship

Consider a rectangular section of width 'b' & depth 'd'.

we know that $\frac{M}{I} = \frac{E}{R} = \frac{\sigma}{y}$.



Section

When stress is less than f_y in the section

$$M = \frac{EI}{R}.$$

$$\therefore \text{Curvature} = \frac{1}{R} = \frac{M}{EI}$$

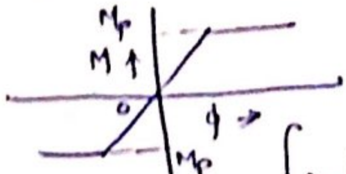
ie; curvature is proportional to moment

When stress @ extreme fibres reaches f_y

$M = M_y = f_y \frac{bd^2}{6}$

$\therefore$ Curvature $= \frac{1}{R} = \frac{M_y}{EI}$.

ii) When yielding is upto y_0 from extreme fibres, the central portion $d-2y_0$ will be within elastic range



$[\therefore K = \frac{y_0}{d}]$

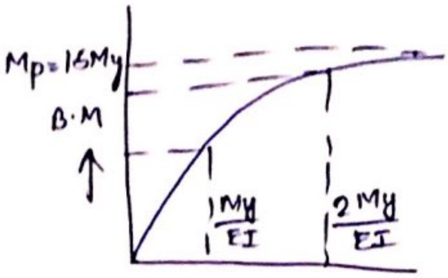
$\frac{E}{R} = \frac{f_y}{\frac{d}{2} - y_0}$

Curvature $= \frac{1}{R} = \frac{f_y}{E(\frac{d}{2} - y_0)} = \frac{f_y}{Ed(\frac{1}{2} - K)}$

But ~~$M = \frac{f_y b}{6}[d^2 + 2dy_0 - 2y_0^2]$~~

~~$= M_y(1 + 2K - 2K^2)$~~

~~curvature $\frac{1}{R}$ = $\frac{6M}{}$~~



Curvature $\frac{1}{R} = \frac{2M_y}{EI}$ & $M = \frac{11 M_y}{8}$ when $y_0 = d/4$, $K = 1/4$.

$\frac{1}{R} = \infty$ & $M = M_p = f_y \frac{bd^2}{4} = 1.5 M_y$ when $y_0 = d/2$, $K = 1/2$

" once plastic hinge is formed rotation takes place freely without any change in moment acting (M_p) in section.

Collapse Load

→ A structure is said to have collapsed structure if the entire structure & part of structure starts undergoing unlimited deformation.

→ This happens when no. of independant static equilibrium equations available are more than the no. of reaction components.

→ state @ which this condition develops is said to be collapse Mechanism & load carried @ this state called collapse load

→ This collapse load is called ultimate Load carrying capacity of structure

→ Determining the collapse load of structure is called plastic analysis.

→ structures permitted to carry only a fraction of collapse loads called working Loads

→ Relationship b/w collapse load & working load is

Collapse load = Load factor × working load

→ Load factor = Ratio of collapse load / working load.

F S = yield stress / working stress

Theorems for finding collapse loads

1) static theorem (lower bound theorem)
2) Kinematic theorem (upper bound theorem)
3) uniqueness theorem (combined theorem)

1) Static theorem

Statement: For a given structure & loading, if there exists a distribution of BM throughout the structure which is both safe & statically admissible with a set of loads W, the value of W must be less than or equal to collapse load Wc.

$W \leq W_c$.

Statically admissible means BM diagram satisfies static equilibrium condition.

Safe means @ no point BM is more than plastic moment capacity (Mp) of section.

→ Lower bound theorem bcoz the values of loads obtained are always less than or equal to collapse load.

2) Kinematic theorem

Statement: For a given structure subject to set of loads W, the value of W found to correspond to any assumed mechanism must be either greater or equal to collapse Load Wc.

$W \geq W_c$.

→ upper bound theorem bcoz, the value of W obtained is greater than or equal to collapse load.

3) uniqueness theorem

Statement: If for a given structure & loading atleast one safe & statically admissble BM distribution can be found and in this distribution, the BM is equal to the fully plastic moment @ enough c/s to cause failure of structure due to unlimited rotations @ plastic hinges, the corresponding Load will be equal to collapse load W_c.

The static & kinematic theorems can be combined to form a theorem which gives unique value for collapse load. This theorem is called uniqueness theorem.

Ultimate strength of fixed & continuous beams

Based on uniqueness theorem two methods of analysis

1) Static method
2) Kinematic method.

1) Static method

→ Suitable for analysis of structure for which shape of Bm's is easily known.

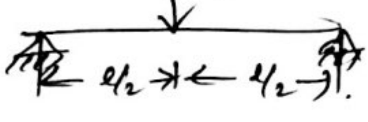
→ only Beams are solved by this method

→ method consists of drawing statically admissable Bm diagr. & equating BM @ sufficient points to plastic moment, so that collapse mechanism forms

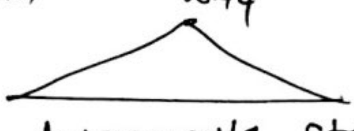
p/o

1) Determine collapse load (W_c) in SSB as shown in fig)

sol



BMD for beam



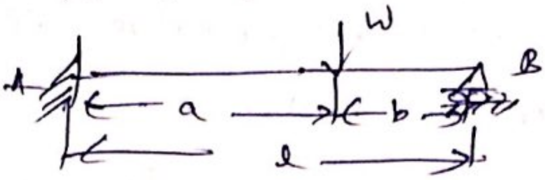
since SSB is a determinate structure, formation of one hinge in beam creates collapse mechanism. Since moment is max under the load the hinge will form @ that place.

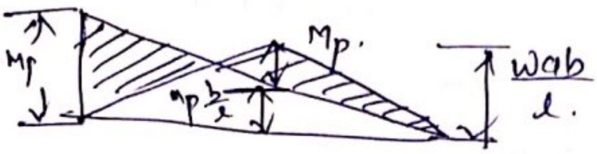
$$\frac{Wl}{4} = M_p.$$

with this, BM is both statically permissible & safe and hinge formation takes place @ sufficient no. of points to develop collapse mechanism

$$\boxed{W_c = \frac{4M_p}{l}}$$

2) Determine collapse load for propped cantilever





Since positive BM is max under load, naturally inner hinge will develop @ this point.

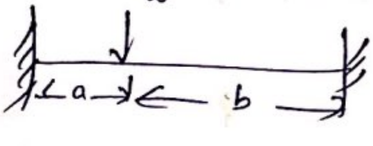
At collapse condition, BM @ A and under the load will equal to mp.

$$\frac{W_c ab}{l} = M_p + M_p\left(\frac{b}{l}\right)$$

$$M_p\left(\frac{l+b}{l}\right) = \frac{W_c ab}{l}.$$

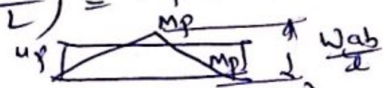
$$\boxed{W_c = M_p\left(\frac{l+b}{ab}\right)}$$

3)



$$\boxed{W_c = \frac{8M_p}{l}}$$

$\rightarrow W_c\left(\frac{ab}{L}\right) = M_p + M_p$



$W_c = 2M_p\left(\frac{L}{ab}\right)$

$a = b = L/2$

$$\boxed{W_c = \frac{8M_p}{l}}$$

4)

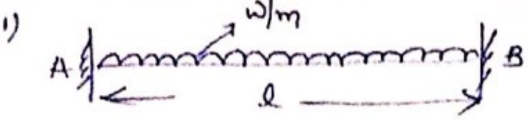
$$\boxed{W_c = 11.655\left(\frac{M_p}{l^2}\right)}$$

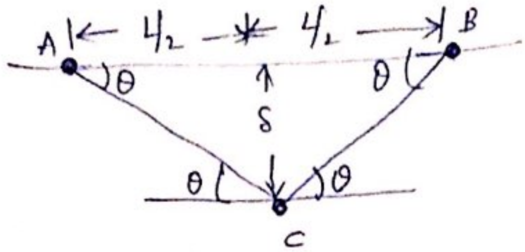
(2) Kinematic method

→ Also known as mechanism method (or) virtual work method

→ Method starts with an assumed collapse mechanism

1)





External workdone

= Total load × Avg distance moved

$= \omega_c L\left(\frac{\delta}{2}\right)$

$= \omega_c L \frac{L}{4}\theta$

$= \left(\frac{\omega_c L^2}{4}\right)\theta$

$\boxed{\delta = \frac{L}{2}\theta}$

$\theta = \frac{\delta}{l/2}$

$\boxed{\delta = \frac{L}{2}\theta}$

Internal work done

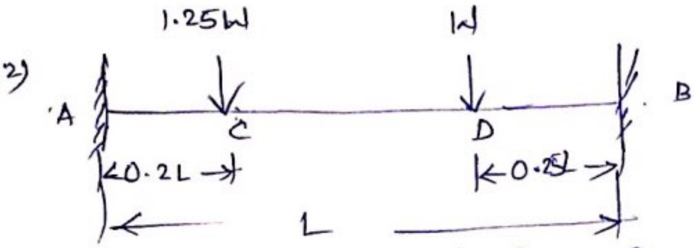
$= M_p\theta + M_p\theta + M_p\theta + M_p\theta$

$= 4M_p\theta$

External workdone = Internal workdone

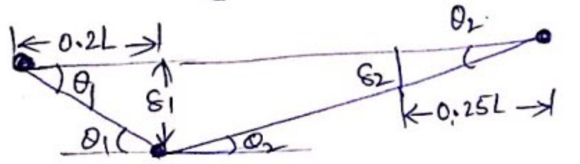
$\left(\frac{W_c L^2}{4}\right)\theta = 4M_p\theta \Rightarrow \boxed{\omega_c = \frac{16M_p}{L^2}}$

2)



At collapse, hinges should form @ supports and another hinge in span.

<u>Mechanism-I</u> (Interior hinge under load 1.25W)



$\theta_1 = \frac{\delta_1}{0.2L}$; $\theta_2 = \frac{\delta_2}{0.25L}$

$\boxed{\delta_1 = 0.2L\theta_1}$; $\boxed{\delta_2 = 0.25L\theta_2}$

$\frac{\delta_1}{0.8L} = \frac{\delta_2}{0.25L}$

$\delta_2 = \left(\frac{0.25L}{0.8L}\right)\delta_1 \Rightarrow \boxed{\delta_2 = \left(\frac{0.25}{0.8}\right)\delta_1} \rightarrow (1)$

$0.2L\theta_1 = 0.8L\theta_2 = \delta_1 \rightarrow (2)$

$\theta_1 = \frac{0.8\,\theta_2}{0.2}$

$\boxed{\theta_1 = 4\theta_2}$

Internal workdone $= M_p\theta_1 + M_p\theta_1 + M_p\theta_2 + M_p\theta_2$

$= 2M_p\theta_1 + 2M_p\theta_2$

$= 2M_p(\theta_1 + \theta_2)$

$= 2M_p(4\theta_2 + \theta_2) = 10M_p\theta_2$

External workdone $= 1.25 W_c \delta_1 + W_c \delta_2$

$$= 1.25 W_c \delta_1 + W_c \left(\frac{0.25}{0.8}\right)\delta_1$$

$$= W_c \left(1.25 + \frac{0.25}{0.8}\right) 0.8 L \theta_2$$

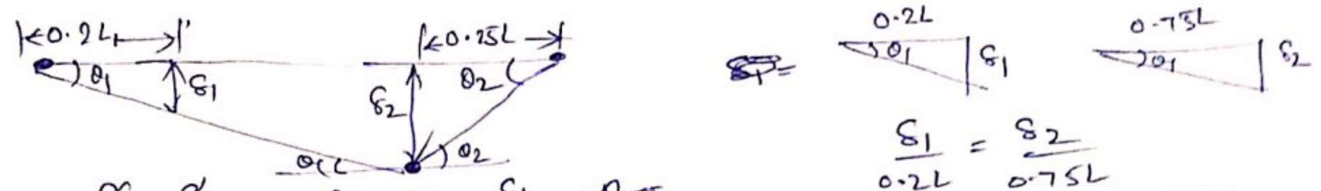
$$= W_c \times 1.25 \times L\theta_2$$

∴ Equating;

$$10 M_p \theta_2 = W_c (1.25) L \theta_2$$

$$\boxed{W_c = \frac{8 M_p}{L}}$$

Mechanism II (Internal hinge under load W)



$\theta_1 = \frac{\delta_1}{0.2L}$; $\theta_1 = \frac{\delta_1}{0.2L}$

$$\frac{\delta_1}{0.2L} = \frac{\delta_2}{0.75L}$$

$$\boxed{\delta_1 = \left(\frac{0.2}{0.75}\right)\delta_2}$$

$$0.75 L \theta_1 = 0.25 L \theta_2 = \delta_2$$

$$\theta_2 = \left(\frac{0.75}{0.25}\right)\theta_1 \Rightarrow \boxed{\theta_2 = 3\theta_1}$$

Internal workdone $= M_p \theta_1 + M_p \theta_1 + M_p \theta_2 + M_p \theta_2$

$$= 2M_p(\theta_1 + \theta_2)$$

$$= 2M_p(\theta_1 + 3\theta_1)$$

$$= 8 M_p \theta_1$$

External workdone $= 1.25 W_c \delta_1 + W_c \delta_2$

$$= 1.25 W_c \left(\frac{0.2}{0.75}\right)\delta_2 + W_c \delta_2$$

$$= W_c \left(1.25 \times \frac{0.2}{0.75} + 1\right)\delta_2$$

$$= W_c \left(1.25\left(\frac{0.2}{0.75}\right) + 1\right) 0.75 L \theta_1$$

$$= W_c L \theta_1$$

Equating

$$8 M_p \theta_1 = W_c L \theta_1$$

$$\boxed{W_c = \frac{8 M_p}{L}}$$

External workdone $= 1.25 W_c \delta_1 + W_c \delta_2$

$$= 1.25 W_c \delta_1 + W_c \left(\frac{0.25}{0.8}\right)\delta_1$$

$$= W_c \left(1.25 + \frac{0.25}{0.8}\right) 0.8 L \theta_2$$

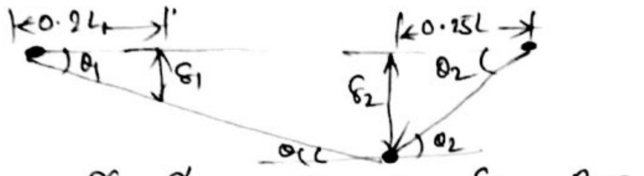
$$= W_c \times 1.25 \times L\theta_2$$

Equating;

$$10 M_p \theta_2 = W_c (1.25) L\theta_2$$

$$\boxed{W_c = \frac{8 M_p}{L}}$$

Mechanism II (Internal hinge under load W)



$\theta_1 = \frac{\delta_1}{0.2L}$ → θ_1 $\quad \theta_1 = \frac{\delta_1}{0.2L}$

$$\frac{\delta_1}{0.2L} = \frac{\delta_2}{0.75L}$$

$$\boxed{\delta_1 = \left(\frac{0.2}{0.75}\right)\delta_2}$$

$$0.75L\,\theta_1 = 0.25L\,\theta_2 = \delta_2$$

$$\theta_2 = \left(\frac{0.75}{0.25}\right)\theta_1 \Rightarrow \boxed{\theta_2 = 3\theta_1}$$

Internal workdone $= M_p\theta_1 + M_p\theta_1 + M_p\theta_2 + M_p\theta_2$

$$= 2M_p(\theta_1 + \theta_2)$$

$$= 2M_p(\theta_1 + 3\theta_1)$$

$$= 8M_p\theta_1$$

External workdone $= 1.25 W_c \delta_1 + W_c \delta_2$

$$= 1.25 W_c \left(\frac{0.2}{0.75}\right)\delta_2 + W_c\delta_2$$

$$= W_c\left(1.25 \times \frac{0.2}{0.75} + 1\right)\delta_2$$

$$= W_c\left(1.25\left(\frac{0.2}{0.75}\right) + 1\right) 0.75 L\theta_1$$

$$= W_c L\theta_1$$

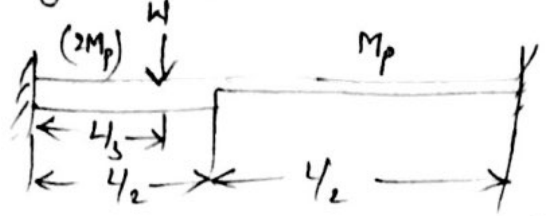
Equating

$$8M_p\theta_1 = W_c L\theta_1$$

$$\boxed{W_c = \frac{8M_p}{L}}$$

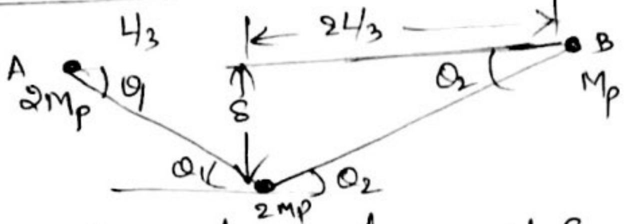
3) Determine collapse load in fixed beam, in which plastic moment Capacity is 2 Mp in one half and Mp in the other half.

Sol



At collapse, plastic hinges will form @ the ends and the third hinge may form under the load.

Mechanism I:



$$\frac{\delta}{L/3} = \theta_1$$

$$\delta = \left(\frac{L}{3}\right)\theta_1$$

$$\delta = \left(\frac{2L}{3}\right)\theta_2$$

External work done $= W_c\,\delta$

Internal work done $= 2M_p\theta_1 + 2M_p\theta_1 + 2M_p\theta_2 + M_p\theta_2$

$$\left(\frac{L}{3}\right)\theta_1 = \left(\frac{2L}{3}\right)\theta_2$$

$$\boxed{\theta_1 = 2\theta_2}$$

$= 2M_p(2\theta_2) + 2M_p(\theta_1 + \theta_2) + M_p\theta_2$

$= 4M_p\theta_2 + 2M_p(3\theta_2) + M_p\theta_2$

$= 11M_p\theta_2$

∴ External work done = Internal work

$$W_c\,\delta = 11M_p\theta_2$$

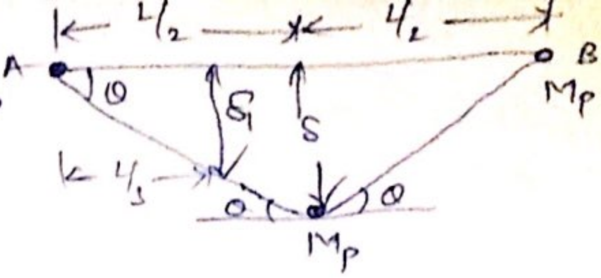
$$W_c\left(\frac{2L}{3}\right)\theta_2 = 11M_p\theta_2$$

$$W_c = \frac{33M_p}{2L}$$

$$\boxed{W_c = \frac{16.5M_p}{L}}$$

Mechanism-II

It is having hinge @ Mid span. Let virtual displacements @ this point be Δ and the rotations @ ends be θ_1 and θ_2 respectively from fig;



$\frac{\delta}{(L/2)} = \theta \Rightarrow \delta = \frac{L}{2}\theta_1 = \frac{L}{2}\theta_2$

$$\boxed{\delta = \frac{L}{2}\theta}$$

Displacement under load $\delta_1 \Rightarrow$ $\frac{\delta_1}{L/3} = \frac{\delta}{L/2}$

$\Rightarrow \delta_1 = \frac{\left(\frac{L}{3}\right)\delta}{\left(\frac{L}{2}\right)} = \frac{2}{3}\delta = \frac{2}{3}\frac{L}{2}\theta$

$$\therefore \boxed{\delta_1 = \frac{L}{3}\theta}$$

Internal work done $= 2M_p\theta + M_p\theta + M_p\theta + M_p\theta$

$= 5M_p\theta$

External work done $= W_c\delta_1$

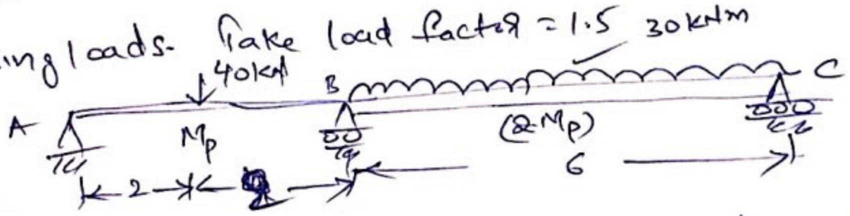
$= W_c\left(\frac{L}{3}\theta\right)$

$5M_p\theta = W_c\left(\frac{L}{3}\theta\right)$

$$\boxed{W_c = \frac{15M_p}{L}} \quad \text{— (2)}$$

from (1) and (2), we conclude that Mechanism-II is the Real Mechanism, and collapse load $\boxed{W_c = \frac{15M_p}{L}}$

4) Calculate the plastic moment capacity req. for continuous beam with working loads. Take load factor = 1.5



sol

Given working loads of 40 kN and 30 kN/m. change the loads into collapse loads

i) Collapse load = working load × FOS

$= 40 \times 1.5$

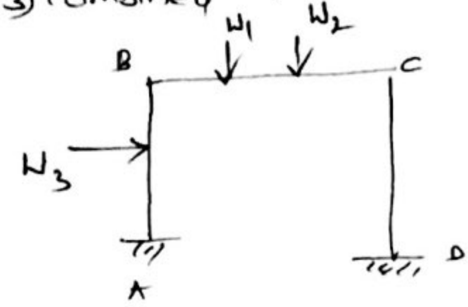
$= 60$ kN.

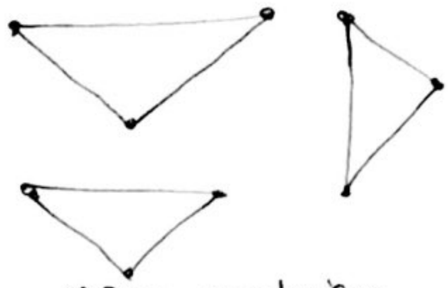
ii) collapse load $= 20 \times 1.5 = 30$ kN/m

Frames (kinematic applied)

The frame may fail by one of the following mechanism

1) Beam Mechanism
2) Sway "
3) Combined "





a) Beam mechanism

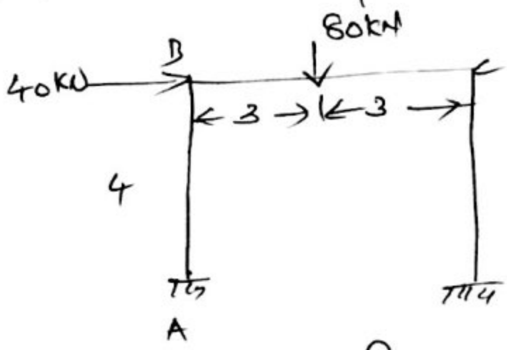
b)

Sway Mechanism

c)

Combined mechanism

(P/o) 1) Determine the plastic moment capacity of a section req. for the frame shown in fig. The loads shown are working loads. Take load factor $\lambda = 1.75$. Assume plastic moment capacity for all members?



Sol

Given frame, change into collapse loads

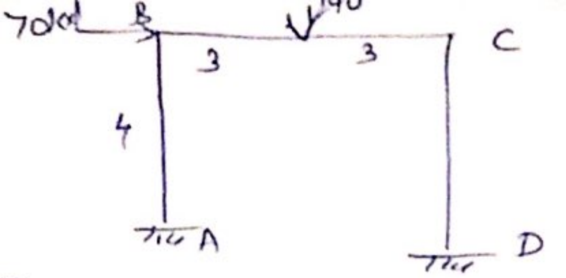
1) collapse load = load factor × working loads

$= 1.75 \times 40$

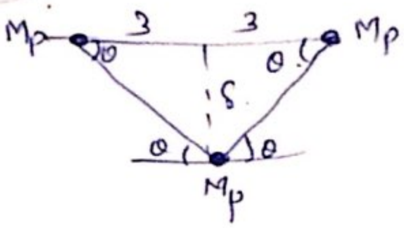
$= 70kN$

2) Collapse load = 1.75×80

$= 140kN$



i) Beam Mechanism



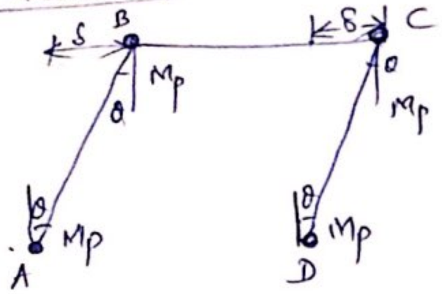
External work done = 140δ
= 140(3θ)

Internal work done = 4Mpθ

$4M_p\theta = 140(3\theta)$

$M_p = 105\,kNm$ — (1)

ii) Sway Mechanism



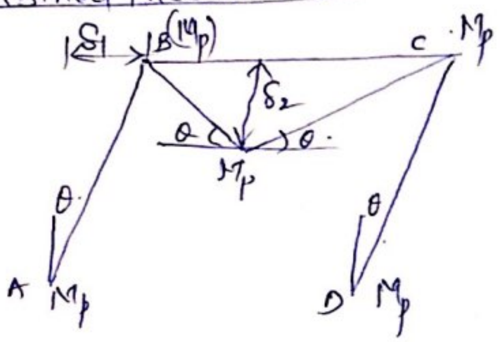
External workdone = 70δ
= 70(4θ)

Internal work done = 4Mpθ.

$4M_p\theta = 70(4\theta)$

$M_p = 70\,kNm$ — (2)

iii) Combined Mechanism



External work done = $70\delta_1 + 140\delta_2$

$= 70(4\theta) + 140(3\theta)$

$= 280\theta + 420\theta$

$= 700\theta$

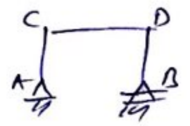
Internal work done = $6M_p\theta$

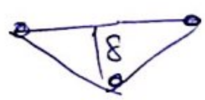
$6M_p\theta = 700\theta$

$M_p = 116.67\,kNm$ — (3)

From (1), (2) and (3): $M_p = 116.67\,kNm$ → Real mechanism

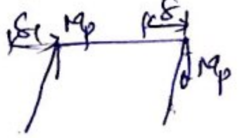
(2) In the frame shown above, if support A and D are hinged Determine plastic moment capacity required!





sol i) Beam mechanism:
(same) $M_p = 105\,kNm$

ii) Sway mechanism



$2M_p\theta = 70(4\theta)$

$M_p = 140\,kNm$

iii) Combined

$4M_p\theta = 700\theta$

$M_p = \frac{700}{4} = 175\,kN$